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Risk-switching insolvency models

Abstract

This paper concerns the Sparre Andersen model with a risk-switching mechanism which enables effective modelling of an insurer's claims. The distributions of the claims' amounts and/or respective waiting times are driven by a Markov chain and the insurer can fit the premium rate in response. The risk-switching methodology generalizes some well-known approaches in the ruin theory, which enables us to treat numerous discrete- and continuous-time models simultaneously and in a unified way. An upper bound for ruin probabilities in a risk-switching setting is also investigated.

Keywords: risk operators, risk-switching models, ruin probabilities, upper bounds, Markov chains

1. Introduction

Regime-switching techniques are currently widespread throughout the actuarial and financial literature[3]. Recently, considerable attention has been paid

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3 To list only a few recent monographs and papers: L. Xu., L. Zhang, D. Yao, *Optimal investment and reinsurance for an insurer under Markov-modulated financial market*, "Insurance: Mathematics and Economics" 2017, vol. 74, pp. 7–19; G. Wang, G. Wang, H. Yang, *On a multi-dimensional risk model with regime switching*, "Insurance: Mathematics and Economics" 2016, vol. 68, pp. 73–83; A. Chen, Ł. Delong, *Optimal investment for a defined-contribution pension scheme under a regime-switching model*, "ASTIN Bulletin" 2015, vol. 45, pp. 397–419; D. Landriault, B. Li, S. Li, *Analysis of a drawdown-based regime-switching Lévy insurance model*, "Insurance: Mathematics and Economics" 2015, vol. 60, pp. 98–107; X. Chen, T. Xiao, X. Yang, *A Markov-modulated jump-diffusion risk model with randomized observation periods and threshold dividend strategy* "Insurance: Mathematics and Economics" 2014, vol. 54, pp. 76–83; A. Guillou, S. Loisel, G. Stupfler, *Estimation of the parameters of a Markov-modulated loss process in insurance*, "Insurance: Mathematics and Economics" 2013, vol. 53, pp. 388–404; S. Asmussen, H. Albrecher, *Ruin probabilities*, 2nd ed., World

to investigating the Markov-modulated Cramér–Lundberg model and its extensions[4]. We will show that the idea of regime-switching, applied to the Sparre Andersen model, leads to a fairly general notion of risk operator which enables one to prove iterative upper and lower bounds for ruin probabilities[5]. Iterating the risk operator can find some applications in the context of insolvency risk management based on Solvency II principles. We refer the reader to Section 4 of Gajek and Rudź[6] where a simulation study is given showing how the Solvency Capital Requirement (SCR) can be determined in accordance with Solvency II regulations.

Set $\mathbb{N}=\{1,2,3,\ldots\}$, $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{N}^0=\mathbb{N}\cup\{0\}$, $\mathbb{N}^1=\mathbb{N}\setminus\{1\}$, $\mathbb{R}_+=(0,\infty)$, $\mathbb{R}_+^0=[0,\infty)$ and $\overline{\mathbb{R}}_+=(0,\infty]$. We assume that all the considered stochastic objects are defined on a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let a random variable X_k denote the amount of the kth claim, T_1 – the moment when the first claim arrives and T_k – the time between the $(k-1)$th claim and the kth one, $k\in\mathbb{N}^1$. Let A_n be the moment when the nth claim arrives. Obviously, $A_n=T_1+\ldots+T_n, n\in\mathbb{N}^0$, with $A_0=0$. A random variable C_k will denote the insurance premium rate during the interval $[A_{k-1},A_k)$. We assume that all the random variables C_k, T_k and X_k are positive (a.s.), $k\in\mathbb{N}$, and that their distributions have no singular parts. Let us denote by $\{I_k\}_{k\in\mathbb{N}^0}$ a time-homogeneous Markov chain with a finite state space $S=\{1, 2,\ldots,s\}$, an initial distribution $(p_i)_{i\in S}$ with positive probabilities $p_i=\mathbb{P}(I_0=i)$, $i\in S$, and a transition matrix $P=(p_{ij})_{i,j\in S}$ with non-negative probabilities $p_{ij}=\mathbb{P}(I_{k+1}=j\,|\,I_k=i)$, $i,j\in S$. The jump from I_{k-1} to I_k can update (if $i\neq j$) the distribution of T_k and/or X_k at the moment A_k only[7]. We can thereby interpret $\{I_k\}_{k\in\mathbb{N}^0}$ as 'switches'.

Let c be a known positive function defined on S. From now on, we make the assumption that the insurance premium rate C_k equals $c(I_{k-1})$. Set $Z_k=$

Scientific, Singapore 2010; S. Asmussen, *Risk theory in a Markovian environment*, "Scandinavian Actuarial Journal" 1989, pp. 69–100; J.M. Reinhard, *On a class of semi-Markov risk models obtained as classical risk models in a Markovian environment*, "ASTIN Bulletin" 1984, vol. 14, pp. 23–43.

[4] For details, see, e.g., H. Albrecher, J. Ivanovs, *A risk model with an observer in a Markov environment*, "Risks" 2013, vol. 1(3), pp. 148–161.

[5] See also L. Gajek, M. Rudź, *A generalization of Gerber's inequality for ruin probabilities in risk-switching models*, "Statistics and Probability Letters" 2017, vol. 129, pp. 236–240; L. Gajek, M. Rudź, *Banach Contraction Principle and ruin probabilities in regime-switching models*, "Insurance: Mathematics and Economics" 2018, vol. 80, pp. 45–53; L. Gajek, M. Rudź, *Finite-horizon ruin probabilities in a risk-switching Sparre Andersen model*, "Methodology and Computing in Applied Probability" 2018, accepted for publication.

[6] For details, see L. Gajek, M. Rudź, *Finite-horizon…*

[7] For details, see L. Gajek, M. Rudź, *A generalization…*

$= X_k - c(I_{k-1})T_k$, $k \in \mathbb{N}$ and $K_n = Z_1 + \ldots + Z_n$, $n \in \mathbb{N}$, with $K_0 = 0$. We will denote by $u \geq 0$ the insurer's surplus at 0 and by $U_n = U(n,u)$ – at the moment A_n, respectively. The surplus process (risk process) $\{U_n\}_{n \in \mathbb{N}^0}$ is defined then by

$$U(n,u) = u - K_n. \tag{1}$$

The framework described above generalizes[8] numerous models of ruin theory, including: the discrete time risk-switching model, the continuous time risk-switching model with exponentially distributed waiting times, the Sparre Andersen risk model, the classical non-switching Cramér–Lundberg risk model and the non-switching discrete time risk model.

The time of ruin τ is said to be the first moment when the insurer's surplus falls below zero. More precisely

$$\tau = \tau(u) = \inf\,\{n \in \mathbb{N} : U(n,u) < 0\}, \tag{2}$$

where $\inf \varnothing$ means ∞. The conditional probability that $\tau(u) \leq n$, given the state i in the beginning, considered as a function of u, is called the probability of ruin at or before the nth claim. We will denote it by $\Psi_n^i(u)$. Clearly

$$\Psi_0^i(u) = 0, \quad i \in S, u \geq 0. \tag{3}$$

From now on

$$\mathbf{\Psi}_n(u) = \left(\Psi_n^1(u), \ldots, \Psi_n^s(u)\right), \quad n \in \mathbb{N}^0, u \geq 0. \tag{4}$$

Let us denote by F^{ij} (G^{ij}, respectively) the conditional distribution of X_1 (T_1, respectively), given the state i in the beginning and the state j at the moment A_1, see Section 2 for details. Set

$$M^i(r) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_0^\infty e^{-r(c(i)t - x)} dF^{ij}(x)\, dG^{ij}(t), \tag{5}$$

[8] For details, see Section 3.

for all $i \in S$ and $r \in \mathbb{R}$. Assume that there exist[9] positive constants $r_0^1,\dots,r_0^s$ such that

$$M^i\left(r_0^i\right)=1, \tag{6}$$

for each $i \in S$. We call $\left(r_0^1,\dots,r_0^s\right)$ the adjustment vector.

Let $\mathfrak{R}$ denote the set of all measurable functions defined on $\mathbb{R}_+^0$ and taking values in [0, 1]. We will denote by $\mathfrak{R}^s$ the following set: $\{\left(\rho_1,\dots,\rho_s\right):\rho_i \in \mathfrak{R}$ for each $i \in S\}$. Its elements will be written in bold.

Let $\boldsymbol{\rho}=\left(\rho_1,\dots,\rho_s\right)\in \mathfrak{R}^s$. We call $\boldsymbol{L}=\left(\mathrm{L}_1,\dots,\mathrm{L}_s\right):\mathfrak{R}^s \to \mathfrak{R}^s$ the operator generated by the risk process (in short risk operator) if

$$\boldsymbol{L\rho}\left(u\right)=\left(\mathrm{L}_1\boldsymbol{\rho}\left(u\right),\dots,\mathrm{L}_s\boldsymbol{\rho}\left(u\right)\right), \quad u \geq 0, \tag{7}$$

where:

$$\begin{aligned} \mathrm{L}_i\boldsymbol{\rho}\left(u\right)=&\sum_{j=1}^{s}p_{ij}\int_0^\infty \int_{(0,u+c(i)t]}\rho_j\left(u+c\left(i\right)t-x\right)dF^{ij}\left(x\right)dG^{ij}\left(t\right)\\ &+\sum_{j=1}^{s}p_{ij}\int_0^\infty \int_{u+c(i)t}^\infty dF^{ij}\left(x\right)dG^{ij}\left(t\right), \quad i \in S. \end{aligned} \tag{8}$$

The above risk operator was used in Gajek and Rudź[10] to improve and generalize Gerber's upper bound for finite-horizon ruin probabilities. What is more, by iterating $\boldsymbol{L}$ on any point from a properly chosen metric space, one can approximate[11] the ultimate ruin probability $\Psi^i\left(u\right)=\mathbb{P}\left(\tau\left(u\right)<\infty | I_0=i\right)$.

For every $i \in S$ and $n \in \mathbb{N}$, $r \in \mathbb{R}_+$ such that

$$\mathbb{E}e^{rK_n}<\infty, \tag{9}$$

let us denote

$$b\left(i,n\right)=\sum_{j=1}^{s}p_{ij}\mathbb{E}^{ij}\left(e^{rK_n}\right), \tag{10}$$

where $\mathbb{E}^{ij}\left(e^{rK_n}\right)$ is the conditional expectation of e^{rK_n}, given the state i in the beginning and the state j at the moment A_1, see Section 2 for details. Throughout the paper, we will assume that $n \in \mathbb{N}$ and $r \in \mathbb{R}_+$ are such that (9) holds.

9 For details, see Theorem 2.
10 L. Gajek, M. Rudź, *A generalization…*
11 For details, see L. Gajek, M. Rudź, *Banach Contraction Principle…*

Let a positive constant r_0 (set, e.g., $r_0 = \min\{r_0^i : i \in S\}$) be such that the following inequality holds:

$$M^i\left(r_0\right) \le 1, \tag{11}$$

for each $i \in S$. Let $R_0^i\left(u\right) = e^{-r_0 u}$ and $R_n^i\left(u\right)$ be the ith coordinate of the nth iteration of $\boldsymbol{L}$ on $\boldsymbol{R}_0\left(u\right) = \left(R_0^1\left(u\right),...,R_0^s\left(u\right)\right)$. Then, $\{R_n^i\}_{n\in\mathbb{N}^0}$ is a non-increasing sequence of upper bounds on $\Psi^i, i \in S$. Moreover, for $r \in \left(0, r_0\right)$, it holds

$$\left|R_n^i\left(u\right) - \Psi_n^i\left(u\right)\right| \le e^{-ru}\frac{r_0}{r_0 - r}b\left(i,n\right), \quad i \in S, u \ge 0,$$

which is the main result of the present paper[12].

A comprehensive treatment of Markov additive processes can be found, e.g., in Asmussen[13] or Feng and Shimizu[14]. For the detailed references to the queuing theory, see, e.g., Reinhard[15] or Asmussen[16]. An operator-like approach dates back to Taylor[17] and it was further extended and generalized[18] by Gajek[19] and Gajek and Rudź[20].

2. Auxiliary results

In this section, we briefly sketch some basic facts concerning the risk operator $\boldsymbol{L}$ and the adjustment vector $\left(r_0^1,...,r_0^s\right)$. For any $B \in \mathcal{F}$ and any random variable Y we will write:

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[12] For details, see Theorem 3.

[13] S. Asmussen, *Applied probability and queues*, 2nd ed., Springer, New York 2003.

[14] R. Feng, Y. Shimizu, *Potential measures for spectrally negative Markov additive processes with applications in ruin theory*, "Insurance: Mathematics and Economics" 2014, vol. 59, pp. 11–26.

[15] J.M. Reinhard, op.cit.

[16] S. Asmussen, *Applied probability*…; S. Asmussen, *Risk theory*…

[17] G.C. Taylor, *Use of differential and integral inequalities to bound ruin and queuing probabilities*, "Scandinavian Actuarial Journal" 1976, pp. 197–208.

[18] For details, see Section 2.

[19] L. Gajek, *On the deficit distribution when ruin occurs-discrete time model*, "Insurance: Mathematics and Economics" 2005, vol. 36, pp. 13–24.

[20] L. Gajek, M. Rudź, *Sharp approximations of ruin probabilities in the discrete time models*, "Scandinavian Actuarial Journal" 2013, pp. 352–382; L. Gajek, M. Rudź, *A generalization*…; L. Gajek, M. Rudź, *Finite-horizon*…; L. Gajek, M. Rudź, *Banach Contraction Principle*… and references therein.

$$\mathbb{P}^{i}\left(B\right)=\mathbb{P}\left(B|I_0=i\right),$$

$$\mathbb{P}^{ij}\left(B\right)=\begin{cases}\mathbb{P}\left(B|I_0=i,I_1=j\right) & \text{if} \quad \mathrm{p}_{ij}>0\\ 0 & \text{if} \quad \mathrm{p}_{ij}=0,\end{cases}$$

$$\mathbb{E}^{ij}\left(Y\right)=\begin{cases}\mathbb{E}(Y\mid I_0=i,I_1=j) & \text{if} \quad \mathrm{p}_{ij}>0\\ 0 & \text{if} \quad \mathrm{p}_{ij}=0,\end{cases}$$

$$\mathbb{P}^{i,j,t,x}\left(B\right)=\mathbb{P}\left(B|I_0=i,I_1=j,T_1=t,X_1=x\right),$$

$$H^{ij}\left(t,x\right)=\mathbb{P}^{ij}\left(T_1\le t,X_1\le x\right),$$

where $i,j\in S$ and $t,x\in\mathbb{R}_+$. With the above notation,

$$F^{ij}\left(x\right)=\mathbb{P}^{ij}\left(X_1\le x\right),$$

$$G^{ij}\left(t\right)=\mathbb{P}^{ij}\left(T_1\le t\right),$$

$$\Psi_n^{i}\left(u\right)=\mathbb{P}^{i}\left(\tau\left(u\right)\le n\right), \tag{12}$$

$$\Psi^{i}\left(u\right)=\mathbb{P}^{i}(\tau\left(u\right)<\infty). \tag{13}$$

Let us define $\ell_i:\mathfrak{R}^s\to\mathfrak{R}$ by

$$\ell_i\boldsymbol{\rho}\left(u\right)=\sum_{j=1}^{s}p_{ij}\int_0^{\infty}\int_{\left(0,u+c\left(i\right)t\right]}\rho_j\left(u+c\left(i\right)t-x\right)dF^{ij}\left(x\right)dG^{ij}\left(t\right), \tag{14}$$

where $i\in S$ and $u\ge 0$.
Obviously,

$$\boldsymbol{\ell\rho}=\left(\ell_1\boldsymbol{\rho},\ldots,\ell_s\boldsymbol{\rho}\right)$$

is a linear operator transforming $\mathfrak{R}^s$ into $\mathfrak{R}^s$, where both the symbols on the left-hand side of the above notation are written in bold, while the coordinates of the operator on the right-hand side are not in bold. From now on, we will use the following conventions:

$$\boldsymbol{L}^0\boldsymbol{\rho}\left(u\right)=\boldsymbol{\rho}\left(u\right),\quad \boldsymbol{L}^1\boldsymbol{\rho}\left(u\right)=\boldsymbol{L}\boldsymbol{\rho}\left(u\right)=\left(\mathrm{L}_1\boldsymbol{\rho}\left(u\right),\ldots,\mathrm{L}_s\boldsymbol{\rho}\left(u\right)\right),$$

$$\boldsymbol{\ell}^0\boldsymbol{\rho}\left(u\right)=\boldsymbol{\rho}\left(u\right),\quad \boldsymbol{\ell}^1\boldsymbol{\rho}\left(u\right)=\boldsymbol{\ell\rho}\left(u\right)=\left(\ell_1\boldsymbol{\rho}\left(u\right),\ldots,\ell_s\boldsymbol{\rho}\left(u\right)\right),$$

where $\boldsymbol{\rho} \in \mathfrak{R}^s$ and $u \geq 0$. Note that for all $n \in \mathbb{N}$, $\boldsymbol{\rho} \in \mathfrak{R}^s$ and $u \geq 0$, the following properties hold true:

$$\boldsymbol{L}^n \boldsymbol{\rho}(u) = \left(\mathrm{L}_1 \boldsymbol{L}^{n-1} \boldsymbol{\rho}(u), \ldots, \mathrm{L}_s \boldsymbol{L}^{n-1} \boldsymbol{\rho}(u)\right), \tag{15}$$

and

$$\boldsymbol{\ell}^n \boldsymbol{\rho}(u) = \left(\ell_1 \boldsymbol{\ell}^{n-1} \boldsymbol{\rho}(u), \ldots, \ell_s \boldsymbol{\ell}^{n-1} \boldsymbol{\rho}(u)\right). \tag{16}$$

An important relationship combining $\boldsymbol{\Psi}_{n+1}$, $\boldsymbol{\Psi}_1$ and $\boldsymbol{L}$ is recalled below.

THEOREM 1[21]. Let the following assumptions hold for all $i, j \in S$, $k \in \mathbb{N}^1$ and $t, x \in \mathbb{R}_+$:

A1. The conditional distribution of the random variables $Z_2, \ldots, Z_k$, given $(I_0 = i, I_1 = j, T_1 = t, X_1 = x)$, is the same as the conditional distribution of the random variables $Z_1, \ldots, Z_{k-1}$, given $I_0 = j$;

A2. $H^{ij}(t,x) = F^{ij}(x) G^{ij}(t)$.

Then, for all $n \in \mathbb{N}^0$ and $u \geq 0$,

$$\boldsymbol{\Psi}_{n+1}(u) = \boldsymbol{L}\boldsymbol{\Psi}_n(u) = \boldsymbol{L}^n \boldsymbol{\Psi}_1(u). \tag{17}$$

The next theorem gives a sufficient condition for the existence of $\left(r_0^1, \ldots, r_0^s\right)$.

THEOREM 2[22]. Assume that for every $i \in S$ the set $M_0^i = \{r \geq 0 : M^i(r) < \infty\}$ is right-open,

$$\sum_{j=1}^{s} p_{ij} \int_0^\infty x dF^{ij}(x) < c(i) \sum_{j=1}^{s} p_{ij} \int_0^\infty t dG^{ij}(t),$$

and $P^{ij_0}(X_1 > c(i) T_1) > 0$ for some $j_0 \in S$. Then there exists a vector $\left(r_0^1, \ldots, r_0^s\right)$ with positive coordinates which satisfy (6).

The following lemma will be used to prove the main result of the present paper.

LEMMA 1[23]. Let the assumptions of Theorem 1 hold. Then for all $i \in S$, $n \in \mathbb{N}$, $\boldsymbol{\rho} \in \mathfrak{R}^s$ and $u \geq 0$

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21 For the proof, see L. Gajek, M. Rudź, *Banach Contraction Principle…*

22 For the proof, see ibidem.

23 Lemma 1 and its proof are based on a Ph.D. thesis by M. Rudź, *Wybrane oszacowania prawdopodobieństwa ruiny [Selected estimates of ruin probabilities]*, Institute of Mathematics of Polish Academy of Sciences, Warsaw 2013.

$$\ell_i \boldsymbol{\ell}^{n-1} \boldsymbol{\rho}(u) = \mathrm{L}_i \boldsymbol{L}^{n-1} \boldsymbol{\rho}(u) - \Psi_n^i(u), \tag{18}$$

or, equivalently,

$$\boldsymbol{\ell}^n \boldsymbol{\rho}(u) = \boldsymbol{L}^n \boldsymbol{\rho}(u) - \boldsymbol{\Psi}_n(u). \tag{19}$$

Proof. Note that, by (8), (14) and (17),

$$\boldsymbol{L\rho}(u) = \left(\ell_1 \boldsymbol{\rho}(u) + \Psi_1^1(u), \ldots, \ell_s \boldsymbol{\rho}(u) + \Psi_1^s(u)\right) = \boldsymbol{\ell\rho}(u) + \boldsymbol{\Psi}_1(u),$$

thus (19) holds for $n = 1$. Assume that (19) holds for some $n \in \mathbb{N}$. We shall show that it holds for $n+1$ as well. Indeed, note that, by (8), (14) and Theorem 1, the following equalities hold:

$$\mathrm{L}_i\left(\boldsymbol{\ell}^n \boldsymbol{\rho}(u) + \boldsymbol{\Psi}_n(u)\right) = \ell_i\left(\boldsymbol{\ell}^n \boldsymbol{\rho}(u) + \boldsymbol{\Psi}_n(u)\right) + \Psi_1^i(u) = \ell_i \boldsymbol{\ell}^n \boldsymbol{\rho}(u) + \Psi_{n+1}^i(u),$$

for every $i \in S$. Summing up, $\boldsymbol{L}^{n+1}\boldsymbol{\rho}(u) = \boldsymbol{\ell}^{n+1}\boldsymbol{\rho}(u) + \boldsymbol{\Psi}_{n+1}(u)$. By the induction principle, (19) holds for every $n \in \mathbb{N}$. By (4), (15) and (16), (18) holds as well. ■

3. Examples of the risk operator for some special cases of the risk-switching model

The risk-switching model generalizes several insurance risk models. In the present section, we briefly sketch some of them, paying special attention to the form of the associated risk operator. We hope it will cause no confusion if we use the same notation in each of the models. In the case of discrete time, we will assume that the random variables $X_1, X_2, \ldots$ are non-negative (a.s.).

3.1. The discrete time risk-switching model

Assume that $T_1, T_2, \ldots$ are nonrandom, i.e., there exists a number $m \in \mathbb{R}_+$ such that $\mathbb{P}(T_k = m) = 1$ for each $k \in \mathbb{N}$. We will interpret $T_1, T_2, \ldots$ as fixed time periods equal to m (for instance, quarters), A_k – as the time to the end of the kth period, and X_k – as the sum of the claims in the kth period. Recall that the random variables $X_1, X_2, \ldots$ are assumed to be non-negative. Since the jump from I_{k-1} to I_k can update the distribution of X_k at the moment A_k only, the changeover of the X_k's distribution is possible at the end of the kth period. Let a positive random variable $\gamma_k = \gamma(I_{k-1}) = c(I_{k-1})m$, where γ is a known function defined on S, denote the total amount of premiums in the kth period. Thus, the amount of premiums in the first period, γ_1, given the state i in the beginning, equals a positive real $\gamma(i)$.

With this notation, under the conventions of Section 1, $Z_k = X_k - \gamma(I_{k-1})$ and

$$U(n,u) = u - \sum_{k=1}^{n} Z_k, \quad n \in \mathbb{N}^0, \tag{20}$$

denotes the insurer's surplus at the end of the nth period. Clearly, $\{U_n\}_{n\in\mathbb{N}^0}$ is the corresponding risk process. The time of ruin τ and the probability of ruin up to the end of the nth period, Ψ_n^i, are defined just like in (2) and (12), respectively. The above model is called the discrete time risk-switching model[24]. An important point to note here is the following form of M^i:

$$M^i(r) = \int_{[0,\infty)} e^{-r(\gamma(i)-x)} \sum_{j=1}^{s} p_{ij} dF^{ij}(x), \quad i \in S,\ r \in \mathbb{R}. \tag{21}$$

Surprisingly enough, M^i reduces to an analogous function associated with a non-switching model[25] in which the aggregated claim distribution in each period is a mixture of distributions $F^{i1}, F^{i2}, \ldots, F^{is}$ with the weights $p_{i1}, p_{i2}, \ldots, p_{is}$, respectively. Thus, each positive r_0^i satisfying Equation (6) is the adjustment coefficient for a model without a switch.

In the discrete time risk-switching model,

$$\mathrm{L}_i \boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int_{[0,u+\gamma(i)]} \rho_j(u+\gamma(i)-x)\, dF^{ij}(x) + \sum_{j=1}^{s} p_{ij} \int_{u+\gamma(i)}^{\infty} dF^{ij}(x), \tag{22}$$

for all $i \in S$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_s) \in \mathfrak{R}^s$ and $u \geq 0$. By Theorem 1, if for all $i, j \in S$, $k \in \mathbb{N}^1$, and $x \in \mathbb{R}_+^0$ the conditional distribution of the random variables $Z_2, \ldots, Z_k$, given $(I_0 = i, I_1 = j, X_1 = x)$, is the same as the conditional distribution of the random variables $Z_1, \ldots, Z_{k-1}$, given $I_0 = j$, then (17) holds.

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24 Cf. L. Gajek, M. Rudź, *Finite-horizon…*; L. Gajek, M. Rudź, *Banach Contraction Principle…*

25 For details, see (26).

3.2. A risk-switching model with exponentially distributed waiting times

Let $\mathbf{1}(x>0)=1$ if $x>0$ and 0 otherwise. If $p_{ij}>0$, assume that $G^{ij}(t)=$ $=\left(1-e^{-\lambda_{ij}t}\right)\mathbf{1}(t>0)$, where the scale parameter $\lambda_{ij}>0$ depends on the states i and j of $\{I_k\}_{k\in\mathbb{N}^0}$ at the moments A_0 and A_1, respectively[26]. Under the assumptions of Section 1, combined with the above one, we can write (8) in the following form:

$$\mathrm{L}_i\boldsymbol{\rho}(u)=\sum_{j=1}^{s}p_{ij}\int_0^\infty\lambda_{ij}e^{-\lambda_{ij}t}\int_{(0,u+c(i)t]}\rho_j\left(u+c(i)t-x\right)dF^{ij}(x)dt+\sum_{j=1}^{s}p_{ij}\int_0^\infty\lambda_{ij}e^{-\lambda_{ij}t}\int_{u+c(i)t}^\infty dF^{ij}(x)dt,$$

where $i\in S$, $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_s)\in\mathfrak{R}^s$ and $u\geq 0$. The function $M^i:\mathbb{R}\to\overline{\mathbb{R}}_+$ is defined by

$$M^i(r)=\sum_{j=1}^{s}p_{ij}\int_0^\infty\lambda_{ij}\ e^{-(\lambda_{ij}+rc(i))t}dt\int_0^\infty e^{rx}dF^{ij}(x).$$

The model described above generalizes the classical Cramér-Lundberg one. Indeed, it is sufficient to make the following assumptions: $s=1$; the sequence $\{T_k\}_{k\in\mathbb{N}}$ consists of independent exponentially distributed random variables with the same scale parameter $\lambda>0$; the sequence $\{X_k\}_{k\in\mathbb{N}}$ consists of independent and identically distributed random variables and is independent of $\{T_k\}_{k\in\mathbb{N}}$ and C_k equals a known positive real. Recall that there are several papers and monographs concerning the Cramér-Lundberg model[27].

3.3. The Sparre Andersen risk model

Under the notation of Section 1, assume that: $s=1$; the sequence $\{T_k\}_{k\in\mathbb{N}}$ consists of independent and identically distributed random variables sharing a distribution function G; the sequence $\{X_k\}_{k\in\mathbb{N}}$ consists of independent and identically distributed random variables sharing a distribution function F; $\{X_k\}_{k\in\mathbb{N}}$

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[26] A special case of this model can be found in L. Gajek, M. Rudź, *Banach Contraction Principle…*

[27] To list only a few recent monographs: F. Lundberg, *I. Approximerad Framställning av Sannolikhetsfunktionen. II. Återförsäkring av Kollektivrisker*, Almqvist & Wiksell, Uppsala 1903; H. Cramér, *On the mathematical theory of risk*, Skandia Jubilee Volume, Stockholm 1930; T. Rolski, H. Schmidli, V. Schmidt, J. Teugels, *Stochastic processes for insurance and finance*, Wiley, New York 1999; S. Asmussen, *Ruin probabilities*, World Scientific, Singapore 2000 (reprinted 2001); P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling extremal events for insurance and finance*, corrected 3rd printing, Springer-Verlag, Berlin–Heidelberg 2001; D.C.M. Dickson, *Insurance risk and ruin*, Cambridge University Press, Cambridge 2005.

is independent of $\{T_k\}_{k\in\mathbb{N}}$ and $C_k = c$, where c will denote here a known positive real. The model described above is called the Sparre Andersen risk model[28].

Under the above assumptions, one can define U_n and τ similarly as in (1) and (2), respectively. The probability of ruin at or before the nth claim is defined by

$$\Psi_n(u) = \mathbb{P}(\tau(u) \le n). \tag{23}$$

Fix the distribution functions F and G, and the premium rate c per unit time. A one-dimensional risk operator $\mathrm{L} : \mathfrak{R} \to \mathfrak{R}$ is defined by

$$\mathrm{L}\rho(u) = \int_0^\infty \int_{(0,u+ct]} \rho(u+ct-x)\,dF(x)\,dG(t) + \int_0^\infty \int_{u+ct}^\infty dF(x)\,dG(t),$$

where $\rho \in \mathfrak{R}$ and $u \ge 0$. The function $M : \mathbb{R} \to \overline{\mathbb{R}}_+$ is defined by

$$M(r) = \int_0^\infty \int_0^\infty e^{-r(ct-x)}\,dF(x)\,dG(t).$$

A positive constant r_0 such that $M(r_0) = 1$, if it exists, is called the adjustment coefficient.

3.4. The discrete time risk model without a switch

Under the notation of Subsection 3.1, let us assume that: $s = 1$; the sequence $\{X_k\}_{k\in\mathbb{N}}$ consists of independent and identically distributed random variables sharing a distribution function F. Let $\gamma_k = \gamma$, where γ is a known positive real. This model can be found, e.g., in the papers by Bowers et al.[29], Klugman et al.[30],

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[28] For details, see, e.g., E. Sparre Andersen, *On the collective theory of risk in the case of contagion between the claims*, in: "Transaction XVth International Congress of Actuaries" New York 1957, vol. II, pp. 219–229; T. Rolski et al., op. cit.; D.A. Stanford, F. Avram, A.B. Badescu, L. Breuer, A. da Silva Soares, G. Latouche, *Phase-type approximations to finite-time ruin probabilities in the Sparre Andersen and stationary renewal risk models*, "ASTIN Bulletin" 2005, vol. 35, pp. 131–144.

[29] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, C.J. Nesbitt, *Actuarial Mathematics*, 2nd ed., The Society of Actuaries, Schaumburg 1997.

[30] S.A. Klugman, H.H. Panjer, G.E. Willmot, *Loss models. From data to decisions*, Wiley, New York 1998.

Rolski et al.[31], Gajek[32], Gajek and Rudź[33] and Rudź[34]. It can be interpreted as a special case of the Sparre Andersen model[35].

A one-dimensional risk operator $\mathrm{L}:\mathfrak{R}\to\mathfrak{R}$ is given by

$$\mathrm{L}\rho(u)=\int_{[0,u+\gamma]}\rho(u+\gamma-x)dF(x)+\int_{u+\gamma}^{\infty}dF(x), \tag{24}$$

for all $\rho\in\mathfrak{R}$ and $u\geq 0$. Note that Assumption A1 is fulfilled[36] in the present model. Thus, by Theorem 1,

$$\Psi_n(u)=\mathrm{L}^n\Psi_0(u), \tag{25}$$

where $\Psi_0(u)=0$ for every $u\geq 0$. The risk operator (24) and the property (25) follow from Gajek[37], where the deficit distribution at ruin was investigated using L.

The function $M:\mathbb{R}\to\overline{\mathbb{R}}_+$ is defined by

$$M(r)=\int_{[0,\infty)}e^{-r(\gamma-x)}dF(x). \tag{26}$$

As in Subsection 3.3, a positive constant r_0 such that $M(r_0)=1$, if it exists, is called the adjustment coefficient.

4. An iterative upper bound for ruin probabilities

Assume that there exists a positive constant r_0 such that (11) holds for each $i\in S$. Given $i\in S$ and $u\geq 0$, we will denote

31 T. Rolski et al., op.cit.

32 L. Gajek, op.cit.

33 L. Gajek, M. Rudź, *Sharp approximations…*

34 M. Rudź, *A method of calculating exact ruin probabilities in discrete time models*, "Roczniki Kolegium Analiz Ekonomicznych" [Annals of Collegium of Economic Analyses] 2015, vol. 37, pp. 307–322; M. Rudź, *Precise estimates of ruin probabilities*, "Metody ilościowe w badaniach ekonomicznych" [Quantitative methods in economics] 2015, vol. XVI, no. 2, pp. 80–88.

35 O. Thorin, *Stationarity aspects of the Sparre Andersen risk process and the corresponding ruin probabilities*, "Scandinavian Actuarial Journal" 1975, pp. 87–98.

36 Cf. Subsection 3.1.

37 L. Gajek, op.cit.

$$R_0^i(u) = e^{-r_0 u},$$

and

$$\boldsymbol{R}_0(u) = \left(R_0^1(u),\dots,R_0^s(u)\right).$$

For every $i \in S$, let us define iteratively a sequence $\{R_n^i\}_{n\in\mathbb{N}}$ by

$$R_n^i(u) = \mathrm{L}_i \boldsymbol{R}_{n-1}(u), \tag{27}$$

where

$$\boldsymbol{R}_n(u) = \left(R_n^1(u),\dots,R_n^s(u)\right), \quad n \in \mathbb{N}, u \geq 0.$$

Note that

$$R_n^i(u) = \mathrm{L}_i \boldsymbol{L}^{n-1} \boldsymbol{R}_0(u), \tag{28}$$

or, equivalently,

$$\boldsymbol{R}_n(u) = \boldsymbol{L}^n \boldsymbol{R}_0(u) = \boldsymbol{L}\boldsymbol{R}_{n-1}(u). \tag{29}$$

For each $i \in S$, by the ideas of Gajek[38], one can show that $\{R_n^i\}_{n\in\mathbb{N}^0}$ is a non-increasing sequence of upper bounds on Ψ^i. Therefore,

$$\Psi_k^i(u) \leq \Psi^i(u) \leq R_k^i(u), \quad i \in S, k \in \mathbb{N}^0, u \geq 0. \tag{30}$$

The following theorem is the main result of the paper. Its special cases can be found in Gajek[39] and Rudź[40].

THEOREM 3. Let the assumptions of Theorem 1 hold. Assume that there exists a positive constant r_0 which satisfies (11) for each $i \in S$. Then

$$\left|R_n^i(u) - \Psi_n^i(u)\right| \leq e^{-ru} \frac{r_0}{r_0 - r} b(i,n), \quad i \in S,\ n \in \mathbb{N},\ r \in (0, r_0), u \geq 0, \tag{31}$$

where $b(i,n)$ are defined by (10).

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38 Ibidem.

39 Ibidem.

40 M. Rudź, *Wybrane oszacowania…*

Proof. Let W be a random variable with the following properties:

P1. W is conditionally independent of all the random variables of the model, given $(I_0 = i, I_1 = j, T_1 = t, X_1 = x)$.

P2. W is conditionally independent of all the random variables of the model, given $(I_0 = i, I_1 = j)$.

P3. W is conditionally independent of all the random variables of the model, given $I_0 = j$.

P4. W is independent of I_0 and I_1.

P5. W is independent of $C_1,\ldots,C_n,\ldots,$ $T_1,\ldots,T_n,\ldots$ and $X_1,\ldots,X_n,\ldots$.

P6. $\mathbb{P}(W \le x) = \left(1 - e^{-r_0 x}\right)\mathbf{1}(x > 0)$,

where $i, j \in S$ and $t, x \in \mathbb{R}_+$ in P1-P3.

We will show that the following inequality holds:

$$\mathbb{P}^i\left(W > U(n,u)\right) \ge \ell_i \ell^{n-1} \boldsymbol{R}_0(u), \tag{32}$$

for all $i \in S$, $n \in \mathbb{N}$ and $u \ge 0$.

By the law of total probability,

$$\begin{aligned}
\mathbb{P}^i\left(W > U(1,u)\right) &= \mathbb{P}^i\left(W > u - X_1 + C_1 T_1\right) \\
&\ge \mathbb{P}^i\left(W > u - X_1 + C_1 T_1, X_1 \le u + C_1 T_1\right) \\
&= \frac{1}{p_i}\mathbb{P}\left(W > u - X_1 + C_1 T_1, X_1 \le u + C_1 T_1, I_0 = i\right) \\
&= \frac{1}{p_i}\sum_{j=1}^{s}\mathbb{P}\left(W > u - X_1 + C_1 T_1, X_1 \le u + C_1 T_1, I_0 = i, I_1 = j\right).
\end{aligned}$$

Therefore, (14), Assumptions A2, P2, P4 and P6 imply that $\mathbb{P}^i\left(W > U(1,u)\right)$ equals

$$\begin{aligned}
&\sum_{\{j \in S:\, p_{ij} > 0\}} p_{ij} \frac{\mathbb{P}\left(W > u - X_1 + C_1 T_1, X_1 \le u + C_1 T_1, I_0 = i, I_1 = j\right)}{\mathbb{P}\left(I_0 = i, I_1 = j\right)} \\
&= \sum_{\{j \in S:\, p_{ij} > 0\}} p_{ij}\, \mathbb{P}^{ij}\left(W > u - X_1 + c(i) T_1, X_1 \le u + c(i) T_1\right) \\
&= \sum_{j=1}^{s} p_{ij} \int_0^{\infty} \int_{(0,\, u + c(i)t]} \mathbb{P}\left(W > u_i'\right) dF^{ij}(x)\, dG^{ij}(t) = \ell_i \boldsymbol{R}_0(u),
\end{aligned}$$

where $u_i' = u + c(i)t - x$. Thus, (32) holds for $n = 1$. Assume that (32) holds for some $n \in \mathbb{N}$. We shall show that it holds for $n+1$ as well. Indeed, in much the same way as above, from P1, P3-P5 and Assumptions A1-A2, we deduce, by the law of total probability and the induction assumption, that

$$\mathbb{P}^i\left(W>U(n+1,u)\right)\geq\mathbb{P}^i\left(W>U(n+1,u),X_1\leq u+C_1T_1\right)$$

$$=\sum_{\{j\in S:\,p_{ij}>0\}}p_{ij}\mathbb{P}^{ij}\left(W>U(n+1,u),X_1\leq u+c(i)T_1\right)$$

$$=\sum_{\{j\in S:\,p_{ij}>0\}}p_{ij}\int_0^\infty\int_{(0,u+c(i)t]}\mathbb{P}^{i,j,t,x}\left(W>u_i^{'}-\sum_{k=2}^{n+1}Z_k\right)dF^{ij}(x)dG^{ij}(t)$$

$$=\sum_{\{j\in S:\,p_{ij}>0\}}p_{ij}\int_0^\infty\int_{(0,u+c(i)t]}\mathbb{P}^j\left(W>U(n,u_i^{'})\right)dF^{ij}(x)dG^{ij}(t)$$

$$\geq\sum_{j=1}^{s}p_{ij}\int_0^\infty\int_{(0,u+c(i)t]}\ell_j\boldsymbol{\ell}^{n-1}\boldsymbol{R}_0\left(u_i^{'}\right)dF^{ij}(x)dG^{ij}(t)$$

$$=\ell_i\boldsymbol{\ell}^n\boldsymbol{R}_0(u).$$

By the induction principle, (32) holds for every $n\in\mathbb{N}$.

For $n\in\mathbb{N}$ and $r\in(0,r_0)$, we get, by P5-P6 and the assumptions of Section 1, that $\mathbb{E}e^{-r(U(n,u)-W)}<\infty$. Therefore, for given $n\in\mathbb{N}$, $r\in(0,r_0)$, $i\in S$ and $u\geq0$, by P2, P4 and P6,

$$\mathbb{P}^i\left(W>U(n,u)\right)=\sum_{\{j\in S:\,p_{ij}>0\}}\frac{p_{ij}}{\mathbb{P}(I_0=i,I_1=j)}\mathbb{P}\left(W>U(n,u),I_0=i,I_1=j\right)$$

$$=\sum_{\{j\in S:\,p_{ij}>0\}}\frac{p_{ij}}{\mathbb{P}(I_0=i,I_1=j)}\int_{\{W>U(n,u),I_0=i,I_1=j\}}d\mathbb{P}$$

$$\leq\sum_{\{j\in S:\,p_{ij}>0\}}\frac{p_{ij}}{\mathbb{P}(I_0=i,I_1=j)}\int_{\{W>U(n,u),I_0=i,I_1=j\}}e^{-r(U(n,u)-W)}d\mathbb{P}$$

$$\leq\sum_{\{j\in S:\,p_{ij}>0\}}\frac{p_{ij}}{\mathbb{P}(I_0=i,I_1=j)}\int_{\{I_0=i,I_1=j\}}e^{-r(U(n,u)-W)}d\mathbb{P}$$

$$=\sum_{\{j\in S:\,p_{ij}>0\}}p_{ij}\mathbb{E}(e^{-r(U(n,u)-W)}|I_0=i,I_1=j)$$

$$=e^{-ru}\frac{r_0}{r_0-r}b(i,n). \tag{33}$$

By (32) and (33), it holds

$$\ell_i \ell^{n-1} \boldsymbol{R}_0(u) \leq e^{-ru} \frac{r_0}{r_0 - r} b(i,n).$$

Thus, by (28), (30) and Lemma 1,

$$\left| R_n^i(u) - \Psi_n^i(u) \right| = \ell_i \ell^{n-1} \boldsymbol{R}_0(u) \leq e^{-ru} \frac{r_0}{r_0 - r} b(i,n),$$

which completes the proof. ■

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Przełącznikowe modele ryzyka niewypłacalności

Streszczenie

Artykuł dotyczy modelu Sparre Andersena z możliwością przełączania charakterystyki ryzyka, która umożliwia efektywne modelowanie szkód ubezpieczyciela. Wysokości szkód oraz czasy oczekiwania na nie mają rozkłady zależące od stanu łańcucha Markowa, a ubezpieczyciel może dynamicznie modyfikować składkę, znając historię. Metodologia przełączania ryzyka ubezpieczeniowego uogólnia pewne znane wyniki w teorii ruiny, co umożliwia modelowanie w jednolity sposób zarówno czasu dyskretnego, jak i ciągłego. Rozważane jest także górne oszacowanie prawdopodobieństwa ruiny w modelu przełącznikowym.

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Słowa kluczowe: operatory ryzyka, przełącznikowe modele ryzyka, prawdopodobieństwo ruiny, górne oszacowania, łańcuchy Markowa