Roczniki Kolegium Analiz Ekonomicznych
Zeszyt 32/2013

Maria Ekes

Application of Generalized Owen Value for Voting Games in Partition Function Form

Astract

In the paper we present an application of the generalized Owen value, defined in our former work, for partition function form games. We apply this value to simple games, modeling multicandidate or multioptional voting. We also present an example of application of this concept to measuring the voting power of deputies in the Polish Sejm.

Keywords: cooperative game with coalition structure, game in partition function form, Owen value, voting power.

1. Introduction

Games in partition function form, introduced by Lucas and Thrall in 1963 (Thrall, Lucas, 1963), are meant to describe the presence of some externalities along with cooperation of (economic or political) players. Such externalities can be incorporated into a model by assuming that the payoff of each coalition depends not only on the composition of this coalition but also on the way that all remaining players are organized. Games in partition function form generalize the class of cooperative games (games with transferable utility). Profits or payoffs of players taking part in a cooperative game can be calculated in many different ways – the most popular and important is the Shapley value (Shapley, 1953). The obvious question is how to measure payoffs of players in partition function form games. There is no unique answer for this question. Two different proposals of generalization of the Shapley value to partition function form games we find in papers of Myerson (1977) and Pham Do and Norde (2007). In both papers authors try to adapt Shapley's axioms to games in partition function form but they obtain different results. Papers of Bolger (1989), Macho-Stadler, Perez-Castrillo and Wettstein (2007) or de Clippel and Serrano (2008) present different approach – they formulate sets of axioms which characterize a value of a partition function form game and they find the unique value satisfying those axioms. The value proposed in the paper De Clippel and Serrano (2008) appears to be the same as in Pham Do and Norde (2007), but axiomatizations are different.

Our present paper concerns the special kind of partition function games, i.e. simple games in which the value of characteristic function can only be 0 or 1. We show that such games can model voting among r candidates or options, and that the generalized Owen value (GOV) introduced in (Ekes, 2010) is an adequate method to measure the voting power of players in this case. This is especially important since it allows for incorporating in the model of voting the phenomena of abstention or absence during voting, which are very often and can change the result of voting. The external effect in voting is particularly strong if the quorum is needed. In this case the change of decision of voters who are absent (and the quorum is not achieved) to participate in voting can influence the result even if "new" voters do not vote for or against the bill but only abstain from voting. In the seminal book of Felsenthal and Machover (1998) the importance of taking abstention into account is also underlined. Felsenthal and Machover propose the measure of voting power based on the Banzhaf index, which takes abstention into account by considering tripartitions instead of bipartitions, usually examined in simple games.

The problem of voting for more than two alternatives is also extensively examined in papers of Bolger (1993, 2000, 2002). He introduces a concept of games with n players and r alternatives and proposes the generalization of Banzhaf index and Shapley index. Indices of power for voting games with r alternatives and with precoalitions were also studied in (Winnicka, 2010).

2. Games in partition function form – preliminaries

2.1. Description of the model

The formal model of the game in partition function form (PFFG) contains the following:

- $N = \{1, 2, ..., n\}$ – the set of players,
- $\mathcal{P}$ – the set of all *partitions* of N, where by a partition we understand the division of the set N into nonempty, disjoint subsets,
- $ECL(N) = \{(S, P) : P \in \mathcal{P}, S \subseteq N, S \in P\}$ – the set of all *embedded coalitions;* an embedded coalition specifies the coalition S along with the structure of coalitions formed by other players,
- $v : ECL(N) \to \mathbb{R}$ – the *characteristic function,* that assigns a real number to each embedded coalition; we assume that $v(\emptyset, P) = 0$ for every $P \in \mathcal{P}$.

While considering games in partition function form we take into account all possible partitions of the set of players and we consider the payoff of a given coalition S as a function not only of the coalition itself but also of the whole partition which includes this coalition. An embedded coalition is a pair consisting of a coalition S and some partition which contains this coalition. Embedded coalitions are arguments of a characteristic function v which describes payoffs of players in a PFFG. Therefore $v(S, P)$ with $S \in P$ and $P \in \mathcal{P}$ is the worth of the coalition S when all players are organized according to P. If $v(S, P) \in \{0, 1\}$ for each $(S, P) \in ECL(N)$, then we call such game a *simple game*.

We shall denote by *PFFG(N)* the set of all partition function form games with n players and we denote by (N, v) a partition function form game in PFFG(N) with a characteristic function v.

If a PFFG has a property that for any coalition $S \subseteq N$ and for each pair of embedded coalitions (S, P) and (S, P'), where $P, P' \in \mathcal{P}$ we have $v(S, P) = v(S, P')$, then we say that this is the game *with no externalities* and it is in fact a cooperative game with characteristic function $v^*(S) = v(S, P)$ for any partition P, which includes a coalition S. A game is *with externalities* if and only if the worth of some coalitions depends on the partition containing these coalitions, i.e. there exists at least one coalition S and two partitions P and P' such that $S \in P$, $S \in P'$ and $v(S, P) \neq v(S, P')$.

2.2. Solution

Definition 1. A *solution* of a PFFG or a *value* is a mapping φ:PFFG$(N) \to \mathbb{R}^n$.

According to the definition 1 a solution is a function which assigns to each PFFG a vector with n coordinates. This vector determines the payoff for every player in the game. In the next section we recall the definition of the generalized Owen value (in short GOV) in order to make this paper complete (for more comprehensive analysis of properties of GOV see also (Ekes, 2010)).

3. Generalization of the Owen value for partition function form games

3.1. Owen value for cooperative games with a coalition structure

We consider the model of a cooperative game with a coalition structure, which is given by:

- $N = \{1, 2, ..., n\}$ – the set of players;
- $P = \{P_1, P_2, ..., P_m\}$ – the partition of the set of players, such that $P_i \neq \emptyset$ for $i = 1, 2, ..., m$, $P_i \cap P_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1,...,m} P_i = N$; we call the sets P_i *precoalitions*;
- $M = \{1, 2, ..., m\}$ – the set of precoalitions;
- $v : 2^N \to \mathbb{R}$ – the characteristic function.

This model was introduced by Owen in (Owen 1977). Owen defined there a value, which is the modification of the Shapley value, reflecting the impact of an a priori division of players to their behavior during the game. The Owen value of a player j belonging to the subset P_i is given by the formula:

$$O_j(v,P) = \\ = \sum_{\substack{H\subset M \\ i\notin H}} \sum_{\substack{S\subset P_i \\ j\notin S}} \frac{h!(m-h-1)!s!(p_j-s-1)!}{m!p_j!} \left(v(H\cup S\cup\{j\}) - v(H\cup S)\right),$$

where:
$h = |H|\,,\ m = |M|\,,\ s = |S|$ and $p_i = |P_i|$.

The concept underlying the Owen value is such that in forming the grand coalition some permutations are not allowed. We take into account only permutations which satisfy the condition that players of a given precoalition appear together one by one. It means that we have to set the order of precoalitions first and then the order of players in each precoalition.

3.2. Outline of the construction of generalized Owen value for partition function form games

Since partition function form games incorporate also in their framework the division of the set players into various subsets, our idea was to generalize the concept of Owen value and to calculate the value of PFFG in a similar way. Our idea can be outlined in the following steps:

- choose a partition $P \in \mathcal{P}$;
- construct a *truncated cooperative game* v_P with coalition structure for the chosen partition P;
- calculate the Owen value for the game v_P;
- take an average of all vectors (Owen values) obtained for every partition P and get the GOV.

3.3. Truncated cooperative games v_P for $P \in \mathcal{P}$

To define the generalized Owen value we need to introduce the concept of a truncated game. It is given by the following definition.

Definition 2. Let us take a game $(N,v) \in \mathrm{PFFG}(N)$ and a partition $P \in \mathcal{P}$. We define the *truncated cooperative game* v_P in the following way:

$$v_P(S(H,P_j,T)) = v(S(H,P_j,T), \widetilde{P}(S(H,P_j,T))),$$

for every coalition $S(H, P_j, T)$ of the form

$$S(H, P_j, T) = \left(\bigcup_{h \in H, H \subset M} P_h \right) \cup T,$$

where $T \subset P_j$ and $j \notin H$ and for the partition $\widetilde{P}(S(H, P_j, T))$ defined in the following way

$$\widetilde{P}(S(H, P_j, T)) = \{S(H, P_j, T), P_j - T\} \cup \{P_h\}_{h \notin H}.$$

A truncated game v_P is a cooperative game with the characteristic function defined only for specific coalitions. Coalitions taken into consideration are of the form of a union of certain coalitions from the partition P (which are given by the set H included in the set of all precoalitions M) and a subset of one of remaining coalitions (this is the subset T of some precoalition P_j which is not the member of H). In other words it means that we exclude all coalitions including proper subsets of more than one precoalition. Observe that in order to calculate the Owen value for any cooperative game v with coalition structure we only need to know the value of characteristic function of coalitions of this kind. Note that if we consider any coalition $S(H, P_j, T)$ constructed in the way described above, then the partition $\widetilde{P}(S(H, P_j, T))$ is a single partition which includes this coalition and preserves the structure of the partition P for all coalitions outside $S(H, P_j, T)$.

3.4. Generalized Owen value

Definition 3. For a given game $(N, v) \in \mathrm{PFFG}(N)$ we define a generalized Owen value in the following way

$$GOV(v; \alpha) = \sum_{P \in \mathcal{P}} \alpha_P \cdot O(v_p, P),$$

where $O(v_P, P)$ is the Owen value of a truncated cooperative game v_P with a precoalition structure given by $P \in \mathcal{P}$ and α_P are nonnegative weights such that

$$\sum_{P \in \mathcal{P}} \alpha_P = 1.$$

Note that 3 defines the whole class of values, depending on the weights α_P. We can treat weights α_P as a priori probabilities of occurrence of specific coalition structures $P \in \mathcal{P}$, or a posteriori frequencies of such occurrence. If we do not have any additional information about such probabilities we take $\alpha_P = \frac{1}{|\mathcal{P}|}$ for all $P \in \mathcal{P}$. In this case $\mathrm{GOV}(v; \frac{1}{|\mathcal{P}|})$ is the average of Owen values of truncated games v_P for all partitions $P \in \mathcal{P}$.

4. Modeling the multicandidate (multioptional) voting

4.1. Voting game

We consider the situation where voters, represented by the set N, are supposed to choose one of r candidates or options ($r \leqslant n$). In course of voting voters partition themselves among r (or less) subsets. The voting rule must specify for each partition of voters

$$\{S_1, ..., S_l\},$$

where $l \leqslant r$, whether S_i is a winning or losing coalition with respect to this partition. We model voting among r options by a simple partition function game $v^s(N, P)$. Observe that in fact we are only interested in the value of embedded coalitions (S, P) such that $|P| \leqslant r$, because voters partition themselves during voting to at most r nonempty subsets. Similar model of multicandidate voting game is described in (Bolger, 1986). Bolger proposes a value, which measures the voting power of players in this case. It is based on the value defined in his earlier paper (Bolger, 1983), but modified to avoid the influence of the addition of dummy players. In our paper we propose to apply the generalized Owen value to measure the power of voters. Since we consider simple games, we will call this measure generalized Owen index (GOI)

4.2. The GOI of a voting game

Let $v^s(N, P)$ be a simple voting game, and assume that voters have to choose one of r options. Let $\mathcal{P}_r$ denote the set of all partitions of N, such that for each $P \in \mathcal{P}_r$ we have $|P| \leqslant r$. We define the GOI of $v^s(N, P)$ by

$$GOI(v^s(N, P)) = \frac{1}{|\mathcal{P}_r|} \sum_{P \in \mathcal{P}_r} O(v^s_p, P).$$

Therefore we calculate the GOV of the simple game $v^s(N, P)$ in this case by taking the positive and equal weights only for partitions satisfying the assumption that voters consider the choice of one of r options.

We can interpret this model in the following way: the game $v^s(N, P)$ prescribes a general voting rule. When considering particular voting with a given number r of options, among which voters have to chose one, we apply respective weights to calculate GOI, taking into account only those partitions of voters which are admissible in this case.

5. Example – voting in Polish Sejm

5.1. Description of the example

We shall examine here the voting in the Polish parliament, called the Sejm. When using the framework of simple games we are able to take into account only two alternatives – voting for or against a proposal. Here we allow voters to abstain from voting, so they have three options among which they choose, or to abstain or be absent which makes four options. There are 460 deputies in Sejm, and the current composition is the following:

- PO (Civic Platform) – 206 deputies,
- PiS (Law and Justice) – 138 deputies,
- Ruch Palikota, RP (Palikot's Movement) – 41 deputies,
- PSL (Polish People's Party) – 29 deputies,
- SLD (Democratic Left Alliance) – 26 deputies,
- Solidarna Polska, SP (United Poland) – 17 deputies,
- 3 non-attached deputies.

Most often the voting rule is the simple majority in the presence of at least a half of all deputies (so the quorum is 230 deputies or more), (see Regulamin 2013). First we allow voters to abstain from voting, so we will consider all partitions of the set of voters to at most three nonempty subsets.

Observe that the situation in this case is slightly different from the model given in the section 4. The value of characteristic function of an embedded coalition depends not only on this coalition and the sole partition, but also on the way in which coalitions in the partition have voted. For example if we have a partition {{PO}, {PiS, SP}, {RP, PSL, SLD, SP, non-attached}}, then the value of a coalition {PiS, SP} depends on whether the coalition {PO} has voted (then it is 0) or has abstained from voting (then it is 1). In this case we should rather consider partitions as sequences of subsets of voters as it is done in (Bolger, 1993) or (Bolger, 2000), where they are called arrangements.

5.2. Results

We calculated the value of generalized Owen index for all parties and for non-attached deputies in the Polish Sejm. We considered all possible ordered partitions, or arrangements in Bolger's terminology, of the set of parties (we include non-attached deputies as singleton parties) and, for each partition, we specified the coalition of parties which is winning according to the rules given in (Regulamin 2013). Since we consider now three different decisions of voting

– vote for, vote against or abstain from voting – in each case the quorum is attained and each partition includes at most three nonempty subsets of the set of parties. Similarly as in (Bolger 1993), we will consider pivots with respect to each alternative ("yes", "no" or "abstain"), which corresponds to the situation that we build grand coalition of voters to vote for the bill, then to vote against the bill and next to abstain. Finally we calculate the value of GOI with respect to each alternative.

In the Table 1 we present the value of GOI of parties in case of voting with abstention, compared to the Shapley-Shubik index of the same players in case without abstention. We denote GOI with respect to existing alternatives by $\mathrm{GOI}_{\mathrm{yes}}$,$\mathrm{GOI}_{\mathrm{no}}$, $\mathrm{GOI}_{\mathrm{abstain}}$ respectively. The value $\mathrm{GOI}_{\mathrm{mean}}$ is the mean value of $\mathrm{GOI}_{\mathrm{yes}}$,$\mathrm{GOI}_{\mathrm{no}}$ and $\mathrm{GOI}_{\mathrm{abstain}}$.

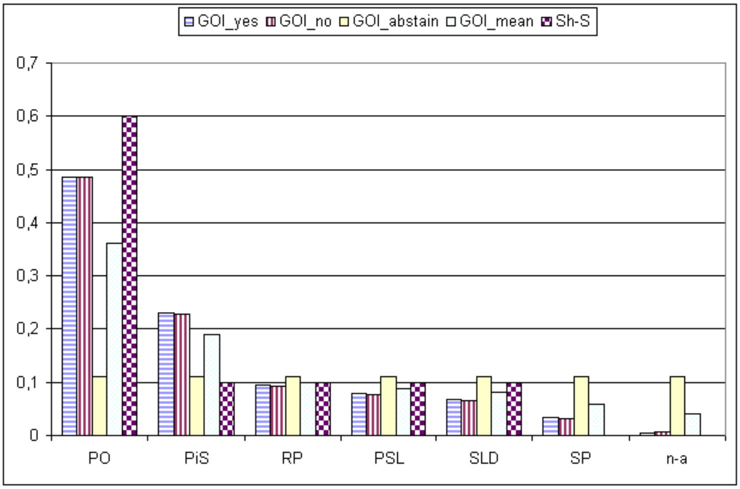
Table 1. GOI and Shapley-Shubik index of parties in the Sejm – the case with three options

	$\mathrm{GOI}_{\mathrm{yes}}$	$\mathrm{GOI}_{\mathrm{no}}$	$\mathrm{GOI}_{\mathrm{abstain}}$	$\mathrm{GOI}_{\mathrm{mean}}$	Sh-S
PO	0.48555	0,48457	0.11	0.36041	0.6
PiS	0.22933	0.22787	0.11	0.18944	0.1
RP	0.09382	0.09287	0.11	0.09927	0.1
PSL	0.0786	0.07662	0.11	0.08878	0.1
SLD	0.06695	0.06599	0.11	0.08135	0.1
SP	0.03298	0.03081	0.11	0.05830	0
n-a	0.00426	0.00709	0.11	0.04082	0

It is rather obvious that when considering the index $\mathrm{GOI}_{\mathrm{abstain}}$, we deal with the unanimity voting in fact (only the value of grand coalition is equal to 1), so the power of all voters is the same. What may be surprising at a first glance is that indices $\mathrm{GOI}_{\mathrm{yes}}$ and $\mathrm{GOI}_{\mathrm{no}}$ are slightly different. It follows from the rule in (Regulamin 2013), which states that the bill passes if there are more votes for than against. It allows for avoiding ties and makes rules concerning voting for and voting against non symmetric.

We present our results also at the figure 1. We observe that including abstention into consideration effects in flattening the results. The Shapley-Shubik index of the biggest party is six times greater than Shapley-Shubik index of the next, while the ratio of their weights is about 1.5. The ratio of the value of their generalized Owen index with respect to voting for or voting against is about 2. Moreover, all parties have positive value of GOI with respect to all alternatives, which means that there are no zero-players when considering abstention. It is obvious that even a single player can change the result of voting in case where large parties abstain from voting.

Figure 1. GOI and Shapley-Shubik index of parties in the Sejm – the case with three options



Taking absence as well as abstention into consideration makes our model even more realistic. Now we can also reflect those situations where the number of absent voters can change the result of voting. We calculated the value of $\mathrm{GOI}_{\mathrm{yes}}$ in this case. It is clear that the value of the generalized Owen index with respect to abstention will remain the same as in case with three options, while the value of $\mathrm{GOI}_{\mathrm{absence}}$ will be equal to Shapley-Shubik index. It follows from the fact that the coalition of absent voters becomes winning in cases where it includes more than 230 voters, and from the theorem proven in (Casajus 2009), which concerns the consistency of the Owen value. The results are shown in the table 2.

Table 2. $\mathrm{GOI}_{\mathrm{yes}}$**index of parties in the Sejm – the case of four options compared to the case of three options**

	$\mathrm{GOI}_{\mathrm{yes}}$(four options)	$\mathrm{GOI}_{\mathrm{yes}}$(three options)
PO	0.45693	0.48555
PiS	0.19995	0.22933
RP	0.10101	0.09382
PSL	0.08732	0.0786
SLD	0.07516	0.06695
SP	0.03875	0.03298
n-a	0.01363	0.00426

We see that including not only abstention but also absence into the model has effects on the rise of power of smaller parties. It appears that including new possibilities or options in the model implies that the voting power of parties is much closer to their share of seats than the Shapley-Shubik index is.

6. Summary

In the paper we presented an application of the generalized Owen value to voting games in partition function form. The most important advantage of this concept is that it gives the tool for examining the choice among more than two alternatives. Observe that if we have given any simple game $v^s(N,P)$, this game defines rules of voting in any case of choice among r alternatives, where $2 \leqslant r \leqslant n$. It means that, for a given voting body, generalized Owen index allows for calculating power of voters in various situations of multioptional choice.

The results of our calculations show that the extension of the model results in approaching the actual weights of parties in Sejm by their generalized Owen index. We do not now how general this result is of course, but we find it interesting and worthy of further and more general analysis.

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Zastosowanie uogólnionej wartości Owena w grach prostych w postaci funkcji rozbicia

Streszczenie

Artykuł jest poświęcony zastosowaniu uogólnionej wartości Owena, zdefiniowanej we wcześniejszych badaniach dla gier w postaci funkcji rozbicia. Proponujemy zastosowanie tej wartości w grach prostych w postaci funkcji rozbicia, modelujących głosowania, w których wybiera się jedną z wielu opcji lub jednego z wielu kandydatów. Przedstawiamy także przykład zastosowania tej wartości do mierzenia siły głosu posłów w Sejmie RP.

Słowa kluczowe: gra kooperacyjna z prekoalicjami, gra w postaci funkcji rozbicia, wartość Owena, siła głosu w głosowaniu.

Author:

Maria Ekes, Department of Mathematics and Mathematical Economics, Warsaw School of Economics, Al. Niepodległości 162, 02-554 Warsaw, Poland,

e-mail: maria.ekes@sgh.waw.pl