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Stochastic orders in the Bayesian framework

Summary

We give a review and a cross section of stochastic ordering problems from the Bayesian point of view – the stochastic ordering of posterior distributions, marginal distributions of data and predictive distributions under order assumptions on sampling distributions and prior distributions. The importance for risk theory and application to actuarial problems are commented.

Keywords: usual stochastic order, dispersive order, likelihood ratio order, increasing convex (stop-loss) order, weighted distributions, prior distributions, posterior distributions, predictive distributions, risk theory

1. Introduction

The problem of interest is how stochastic orders of sampling and prior distributions may be transferred to posterior and marginal data distributions and what is the change of the posterior distributions in respect of prior ones from the aspect of stochastic orders. We collect and interpret useful existing results from this point of view. Then we consider predictive distributions – the main Bayesian tool for statistical prediction, by giving some statements derived from previous results. This is of interest for the reliability theory, survival analysis, comparing risks and also for Bayesian robustness as a look at consequences of various choices of prior distributions.[2]

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[2] See M. Męczarski, *Stochastic orders and classes of prior distributions*, "Statistics in Transition" 2004, vol. 6, no. 7, pp. 1121–1129.

Comparing risks is considered by eminent authors as an essential part of actuarial practice.[3] Actuarial risks are represented in the form of random variables and their distributions, so stochastic orders constitute mathematical tools to handle such problems. The stochastic ordering issues are often considered in the insurance risk theory. Bäuerle and Müller[4] establish consistency and bounds for risk measures implied by the usual stochastic and convex orders. Moreover, a large part of the theory of stochastic orders was developed within the risk theory.[5] The implementation for actuarial practice is clear: for example, Heilmann and Schröter[6] give a number of straightforward applications, Denuit and Lefèvre[7] define some stochastic orders for discrete distributions and apply them for bounds of premium or of ruin probabilities.

Denuit et al.[8] enumerate a number of desirable properties for stochastic orderswith respect to their usefulness in comparing risks: stability under mixture, stability under convolution, under compounding and under limit. Our aim is to investigate the stability under the operations on probability distributions which are typical in Bayesian statistic, because of the importance of the Bayesian approach, as well the ideas used as techniques for actuarial issues. These are computing posterior distributions, predictive distributions and marginal distributions of data. The case of the marginal distribution of data is covered by the case of the mixtures, but we treat it as well because of the completeness and indispensability for predictive distributions.

Let us recall the general Bayesian statistical model $(X.M, P=\{P_\theta\}_{\theta\in\Theta})$, where X is a sample space, M – a σ-algebra of events in and P is a family of probability distributions. Moreover, we assume that there exists a probability space (Θ, F, Π), where Π is a prior distribution. Then X is a random sample with the values $x \in X$, $X \mid T=\theta \sim f(\cdot \mid \theta)$, where T is a Θ-valued random variable, $T \sim \Pi$. Now the formula

3 See M. Denuit et al., *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, Wiley, New York 2005 and R. Kaas et al., *Modern Actuarial Risk Theory Using R*, Springer, Berlin-Heidelberg 2008.

4 N. Bäuerle, A. Müller, *Stochastic orders and risk measures: Consistency and bounds*, "Insurance: Mathematics and Economics" 2006, vol. 38, pp. 132–148.

5 Denuit et al., op.cit.

6 W.R. Heilmann, K.J. Schröter, *Orderings of risks and their actuarial applications*, in: *Stochastic Orders and Decisions under Risk*, eds Mosler K., Scarsini M., IMS Lecture Notes – Monograph Series 19, Institute of Mathematical Statistics, Hayward, CA 1991, pp. 157–173.

7 M. Denuit, C. Lefèvre, *Some new classes of stochastic order among arithmetic random variables, with applications in actuarial sciences*, "Insurance: Mathematics and Economics" 1997, vol. 20, pp. 197–213.

8 M. Denuit et al., op.cit., chapter 3.

$$\pi(\theta \mid x) = \frac{f(x \mid \theta)\pi(\theta)}{\int_\Theta f(x \mid \theta)\pi(\theta)d\theta}$$

gives the posterior distribution and

$$m_\pi(x) = \int_\Theta f(x \mid \theta)\pi(\theta)d\theta = E_\pi f(x \mid T)$$

is the marginal density of the sample.

Like many authors of papers on Bayesian statistics we often use the same notation θ for a random variable, for its values and for arguments of prior densities and cumulative distribution functions unless a misunderstanding may occur. Throughout the paper actually $\Theta \subset \mathbf{R}$ and consequently we integrate with respect to the Lebesgue measure unless it is done with respect to the counting measure in case of discrete distributions. For fundamentals and details of Bayesian approach see e. g. Robert's monograph.[9]

2. Usual stochastic and dispersive orders

As we know, stochastic orders are order relations in the set of probability distributions on a fixed probability space or, equivalently, in the set of random variables on this probability space with these distributions. In some economic applications the term "stochastic dominance" is often used. However, a number of stochastic orderings is not included into the scheme of stochastic dominance of successive orders. Basic definitions and concepts are explained in the monographs by Marshall, Olkin and Arnold,[10] Shaked and Shanthikumar,[11] Müller and Stoyan[12] and Denuit et al.[13] We take into consideration some chosen types of orders: the usual stochastic order (defined by magnitude of tail probabilities

[9] C. Robert, *Bayesian Choice*, Second Edition, Springer, New York 2007.

[10] A.W. Marshall, I. Olkin, B. Arnold, *Inequalities. Theory of Majorization and Its Applications*, Second Edition, Springer, New York 2011; earlier edition: A.W. Marshall, I. Olkin, *Inequalities. Theory of Majorization and Its Applications*, Academic Press, New York 1979.

[11] M. Shaked, J.G. Shanthikumar, *Stochastic Orders and Their Applications*, Academic Press, Boston 1994 and M. Shaked, J.G. Shanthikumar, *Stochastic Orders*, Springer, New York 2007.

[12] A. Müller, D. Stoyan, *Comparison Methods for Stochastic Models and Risks*, J. Wiley, Chichester 2002.

[13] Denuit et al., op.cit.

– see below) and the dispersive order (defined by a measure of dispersion of a probability distribution), which seem to be the simplest to interpret and use. Then we move to the likelihood ratio order and in the last section to the increasing convex (stop-loss) order. Below we recall definitions and properties which are necessary hereafter.

Definition 1. Let X and Y be random variables on a fixed probability space with cumulative distribution functions F and G, respectively. The random variable X is said to be less than Y in the usual stochastic order (we write $X \le_{st} Y$), if $(\forall x \in \mathbf{R})\ F(x) \ge G(x)$ or equivalently $1 - F(x) \le 1 - G(x)$.

We see that Definition 1 organises probability distributions by their tail probabilities, i. e. probabilities of large values. It can be proved[14] that Definition 1 is equivalent to the relation $E\varphi(X) \le E\varphi(Y)$ for any nondecreasing function φ such that both sides exist. Observe also that $X \le_{st} Y$ and $Y \le_{st} X$ is equivalent to equal distributions of X and Y. The usual stochastic order is sometimes called the first order stochastic dominance.

In Bayesian statistical analysis order properties for conditional distributions are needed, as follows.

Theorem 1.[15] Let X, Y and T be random variables such that the conditional distributions satisfy the following relation:

$$(\forall \theta \in \Theta)\, X \,|\, T = \theta \le_{st} Y \,|\, T = \theta.$$

Then $X \le_{st} Y$.

The assumption corresponds to the usual stochastic order of sampling distributions. The conclusion is equivalent to $F_m^\pi \le_{st} G_m^\pi$, where the subscript m denotes marginal distributions of observations and the superscript π stresses the dependence of the prior Π.

In terms of mixtures of distributions we can say that the usual stochastic order is closed with respect to mixtures and in terms of Bayesian statistics that the usual stochastic order of sampling distributions may be transferred to marginal distributions of data.

Theorem 2.[16] Let us consider the family of distributions $\{F(\cdot \,|\, \theta), \theta \in \Theta\}$. Let $X(\theta)$ be a random variable with the distribution function $F(\cdot \,|\, \theta)$. For random

14 See A.W. Marshall, I. Olkin, op.cit.

15 M. Shaked, J.G. Shanthikumar, *Stochastic Orders*, Springer, New York 2007.

16 Ibidem.

variables T_i, $i=1,2$, sharing their support included in Θ and with distribution functions Π_i, $i=1,2$, let $Y_i = X(T_i)$ denote random variables with the distribution functions H_i defined by

$$H_i(x) = \int_\Theta F(x \mid \theta) d\Pi_i(\theta), \quad x \in \mathbf{R}.$$

If $X(\theta) \leq_{st} X(\theta')$ for all $\theta, \theta' \in \Theta$ such that $\theta \leq \theta'$ and if $\Pi_1 \leq_{st} \Pi_2$, then $Y_1 \leq_{st} Y_2$.

In terms of Bayesian statistics this means that the stochastic order of prior distributions provided monotonicity of sampling distributions with respect to the stochastic order is transferred to marginal distributions of data.

From the Bayesian point of view questions of interest concern posterior distributions and are as follows:

1) Does the condition $X(\theta) \leq_{st} X(\theta')$ for all $\theta, \theta' \in \Theta$ such that $\theta \leq \theta'$ imply

$$T \mid X = x \leq_{st} T \mid X = x', \text{ where } x \leq x' \,?$$

2) Does the relation $\Pi_1 \leq_{st} \Pi_2$, where $T_i : \Pi_i$, $i=1,2$, imply

$$T_1 \mid X = x \leq_{st} T_2 \mid X = x?$$

These properties are not true, i.e. we cannot say that posterior distributions preserve the usual stochastic order of sampling distributions or of prior distributions (see Section 3).

3. Weighted distributions

The structure of posterior distributions coincides with the structure of weighted distributions.[17] Order properties of the weighted distributions are quite well known (see below).

[17] C.R. Rao, *On discrete distributions arising out of method of ascertainment*, *Sankhyā Ser. A*, 1965, vol. 27, pp. 311–324; G.P. Patil, C.R. Rao, *Weighted distributions and size biased sampling with applications to wildlife populations and human families*, "Biometrics" 1978, vol. 34, pp. 179–189.

Definition 2. Let X be a random variable, F – its cumulative distribution function (cdf) and f – the corresponding density; let w be a nonnegative weight function. We assume $Ew(X)$ to exist. The weighted F distribution under the weight function w is a distribution with the following cumulative distribution function:

$$\tilde{F}_w(x) = \frac{1}{Ew(X)} \int_{-\infty}^{x} w(t) dF(t)$$

and with the density $\tilde{f}_w(x) = \dfrac{w(x)}{Ew(X)} f(x)$.

It is evident that posterior distributions coincide with prior distributions weighted by likelihood functions, i. e. $w(\theta) = f(x \mid \theta)$ for a given $x \in X$. This observation seems to be "suspended" or even undirectly suggested by Shaked and Shanthikumar,[18] but it is not expressed explicitly.

For weighted distributions there exist many results on preserving various stochastic orderings. Usually assumptions on weight functions are required. The usual stochastic order is not preserved under weighting without such assumptions,[19] so, in general, it is not preserved under computing posterior distribution, either.

Let us consider another interesting stochastic ordering, the dispersive order, as follows.

Definition 3.[20] Let X and Y be random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} denote the inverses of the distribution functions, continuous on the right, i. e. $F^{-1}(\alpha) = \inf\{x \in \mathbf{R} : F(x) \geq \alpha\}$. It is said that the variable X is less than Y in the dispersive order (we write $X \leq_{disp} Y$) if and only if $(\forall 0 < \alpha \leq \beta < 1) \;\; F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$.

An equivalent condition is that the function $G^{-1}(F(x)) - x$ is nondecreasing in x. This is a consequence of the fact that the definition means that the function $G^{-1}(y) - F^{-1}(y)$ is nondecreasing with respect to $y \in (0,1)$. We may observe that the dispersive order consists in comparing differences between any pair of quantiles.

The following theorem gives a relationship between the usual stochastic and dispersive orders.

[18] M. Shaked, J.G. Shanthikumar, *Stochastic Orders*, Springer, New York 2007, chapter 1.C.

[19] See J. Bartoszewicz, M. Skolimowska, *Preservation of classes of life distributions and stochastic orders under weighting*, "Statistics and Probability Letters" 2006, vol. 76, pp. 587–596.

[20] M. Shaked, J.G. Shanthikumat, op.cit.

Theorem 3.[21] If the random variables X and Y satisfy the equality $\inf suppX = \inf suppY$ (where $suppX$ means the support of the probability distribution of a random variable X), then $X \leq_{disp} Y$ implies $X \leq_{st} Y$.

The dispersive ordering is not closed in respect of weighting[22] unless assumptions on monotonicity of weight functions are made. However, a likelihood function has a maximum point at a maximum likelihood estimate and there are few statistical models where it can be monotone (models with a parameter as a bound of a probability support). Bartoszewicz[23] proved the following theorem on weighted distributions which may extend such restrictions for the closeness of the dispersive ordering in respect of weighting.

Theorem 4.[24] Let F and G be absolutely continuous. Let F have the DFR property (decreasing failure rate, i. e. the function $\frac{f(x)}{1-F(x)}$ is nonincreasing) and G have the IRFR property (increasing reversed failure rate, i. e. the function $\frac{g(x)}{G(x)}$ is nondecreasing). Let w be a weight function being of the form $w(x) = \varphi(v(x))$, where v is positive decreasing log-convex (i.e. the logarithm of this function is convex) on $A = suppF \cup suppG$ and φ is positive increasing log-convex on the set $v(A)$. If $X \leq_{disp} Y$, then $X_w \leq_{disp} Y_w$.

The usefulness of Theorem 4 to the Bayesian approach depends on whether a typical likelihood function may be presented in the form $l(\theta, x) = \varphi_x(v_x(\theta))$.

Remark. In many useful statistical models likelihood functions come from the exponential family of distributions: $l(\theta, x) = h(x)e^{c(\theta)t(x)-b(\theta)}$. If we take $\varphi_x(z) = h(x)e^z$ and $v_x(\theta) = c(\theta)t(x) - b(\theta)$, then we obtain that the function φ_x is positive, decreasing and log-convex. Properties of v_x depend on the probability distribution of interest. It should be:

(i) $c(\theta)t(x) - b(\theta) > 0$,

(ii) $c'(\theta)t(x) - b'(\theta) \leq 0$,

(iii) $\frac{d^2}{d\theta^2}\log[c(\theta)t(x) - b(\theta)] \geq 0$,

[21] Ibidem.

[22] J. Bartoszewicz, M. Skolimowska, op.cit.

[23] J. Bartoszewicz, *On a representation of weighted distributions*, "Statistics and Probability Letters" 2009, vol. 79, pp. 1690–1694.

[24] Ibidem.

where
(iii) is equivalent to $[c''(\theta)t(x)-b''(\theta)][c(\theta)t(x)-b(\theta)]-[c'(\theta)t(x)-b'(\theta)]^2 \geq 0$.

This may depend also on the observed x. Observe that (i) and (iii) imply $c''(\theta)t(x)-b''(\theta) \geq 0$. Now, it is easy to see that for some important models these inequalities may be contradictive, for example for the normal, Poisson and exponential models.

For a Bayesian robustness approach[25] it would be interesting to construct intervals of distributions generated by stochastic orders. In such a case, comparing pairs with different particular assumptions for a predecessor and a successor is not relevant.

4. The likelihood ratio order

The likelihood ratio order – as below – seems less intuitive than the usual stochastic and dispersive orders. However, the monotone likelihood ratio is a well known assumption to construct tests of one-sided hypotheses and actually may be quite familiar in statistics.

Definition 4.[26] Let X and Y be real random variables with distribution functions F and G, respectively. It is said that the variable X is less than Y in the likelihood ratio (LR) order (we write $X \leq_{LR} Y$) if and only if the ratio $\frac{g(t)}{f(t)}$ is an increasing function of t on the set $suppX \cup suppY$.

We accept the convention $\frac{a}{0} = +\infty$ for $a > 0$.

Remark. Some equivalent conditions for Definition 4 are as follows:

$$(i) \quad A \leq B \Rightarrow \frac{P(Y \in B)}{P(X \in B)} \geq \frac{P(Y \in A)}{P(X \in A)}$$

for any Borel sets A and B, where $A \leq B$ means that $(\forall x \in A, y \in B)\, x \leq y$;

$$(ii) \quad F(\cdot \mid A) \leq_{st} G(\cdot \mid A)$$

for any Borel set A.

[25] See, e.g., M. Męczarski, op.cit.
[26] M. Shaked, J.G. Shanthikumar, op.cit.

The second condition means the usual stochastic order for any pair of conditional distributions under any random event which implies that $F \leq_{LR} G \Rightarrow F \leq_{st} G$ as well.

$$(iii) \quad GF^{-1} \text{ is a convex function.}$$

The third condition is quite easy to prove and it is related to a general way of defining stochastic orders.[27]

For the likelihood ratio order it can be shown that the conclusions of Theorems 1 and 2 hold, i. e. it is closed under mixtures. The likelihood ratio order is closed under weighting for an arbitrary weight function.[28] For posterior distributions explicit precise statements and proofs are as follows.

Theorem 5. If the distribution of X is increasing with respect to the LR order in θ, then the conditional distribution of θ under $X = x$ is increasing in x with respect to the LR order, i. e. if $P_\theta \leq_{LR} P_{\theta'}$ for $\theta \leq \theta'$, then $\Pi(\cdot \mid X = x) \leq_{LR} \Pi(\cdot \mid X = x')$ for $x \leq x'$.

Proof. We have $\pi(\theta \mid x) = \dfrac{f(x \mid \theta)\pi(\theta)}{m_\pi(x)}$. It should be shown that $\dfrac{\pi(\theta \mid x')}{\pi(\theta \mid x)}$ is an increasing function of θ. So let $\theta' > \theta$. We write

$$\frac{\pi(\theta' \mid x')}{\pi(\theta' \mid x)} = \frac{f(x' \mid \theta')\pi(\theta')m_\pi(x)}{m_\pi(x')f(x \mid \theta')\pi(\theta')} = \frac{f(x' \mid \theta')m_\pi(x)}{f(x \mid \theta')m_\pi(x')}.$$

But since $P_\theta \leq_{LR} P_{\theta'}$, we have

$$\frac{f(x' \mid \theta')}{f(x' \mid \theta)} \geq \frac{f(x \mid \theta')}{f(x \mid \theta)},$$

what implies

$$\frac{f(x' \mid \theta')}{f(x \mid \theta')} \geq \frac{f(x' \mid \theta)}{f(x \mid \theta)}.$$

Thus,

$$\frac{\pi(\theta' \mid x')}{\pi(\theta' \mid x)} \geq \frac{f(x' \mid \theta)\pi(\theta)m_\pi(x)}{m_\pi(x')f(x \mid \theta)\pi(\theta)} = \frac{\pi(\theta \mid x')}{\pi(\theta \mid x)}. \qquad \blacksquare$$

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[27] See M. Shaked, J.G. Shanthikumar, op.cit.; E.I. Lehmann, J. Rojo, *Invariance directional orderings*, "The Annals of Statistics" 1992, vol. 20, pp. 2100–2110; M. Frąszczak, J. Bartoszewicz, *Invariance of relative inverse function orderings under compositions of distributions*, "Applicationes Mathematicae" 2012, vol. 39, pp. 283–292.

[28] J. Bartoszewicz, M. Skolimowska, op.cit.

This property means that the LR order of sample distributions is preserved by posterior distributions. It was given by Shaked and Shanthikumar[29] in the form of a remark after Whitt.[30]

Theorem 6. If $\Pi_1 \leq_{LR} \Pi_2$, then $\Pi_1(\cdot \mid X = x) \leq_{LR} \Pi_2(\cdot \mid X = x)$.

Proof. It should be shown that $\dfrac{\pi_2(\theta \mid x)}{\pi_1(\theta \mid x)}$ is a nondecreasing function of θ. Let $\theta' \geq \theta$. Then

$$\frac{\pi_2(\theta' \mid x)}{\pi_1(\theta' \mid x)} = \frac{f(x \mid \theta')\pi_2(\theta')m_{\pi_1}(x)}{m_{\pi_2}(x)f(x \mid \theta')\pi_1(\theta')} = \frac{\pi_2(\theta')m_{\pi_1}(x)}{m_{\pi_2}(x)\pi_1(\theta')},$$

but

$$\frac{\pi_2(\theta')}{\pi_1(\theta')} \geq \frac{\pi_2(\theta)}{\pi_1(\theta)},$$

hence

$$\frac{\pi_2(\theta' \mid x)}{\pi_1(\theta' \mid x)} \geq \frac{f(x \mid \theta)\pi_2(\theta)m_{\pi_1}(x)}{m_{\pi_2}(x)f(x \mid \theta)\pi_1(\theta)} = \frac{\pi_2(\theta \mid x)}{\pi_1(\theta \mid x)}. \qquad \blacksquare$$

This property means that the LR order of prior distributions is preserved by posterior distributions. The result is cited by Shaked and Shanthikumar[31] as a remark after Spizzichino.[32] The version for weighted distribution was shown, as mentioned above, by Bartoszewicz and Skolimowska[33] (cited also by Shaked and Shanthikumar[34]).

The LR order implies the usual stochastic ordering. This means that the usual stochastic order may be transferred onto posterior distributions, under the LR ordering, which is a stronger assumption.

29 M. Shaked, J.G. Shanthikumar, op.cit.

30 W. Whitt, *A note on the influence of the sample on the posterior distribution*, "Journal of American Statistical Association" 1979, vol. 74, pp. 424–426.

31 M. Shaked, J.G. Shanthikumar, op.cit.

32 F. Spizzichino, *Subjective Probability Models for Lifetimes*, Chapman and Hall/CRC, Boca Raton 2001.

33 J. Bartoszewicz, M. Skolimowska, op.cit.

34 M. Shaked, J.G. Shanthikumar, op.cit.

5. Prior and posterior distributions – ordering comparisons

Hereafter we discuss relationship between prior and posterior distribution with respect to the considered stochastic orders. Bartoszewicz and Skolimowska[35] proved the following implication for weighted distributions.

Theorem 7. (1) If the weight function w is increasing, then $F \leq_{LR} \tilde{F}_w$; (2) if w is decreasing, then $\tilde{F}_w \leq_{LR} F$.

However, if we need w as a likelihood function, its monotonicity rarely holds. Let us denote $\Pi^*(\theta) = \Pi(\theta \mid x)$, the posterior cdf. We can use this notation when correspondence to any fixed data x can be omitted. Błażej[36] gave, in terms of weighted distributions, equivalent conditions for orderings, defined by properties of the following function $\breve{\Pi}_x$:

$$\breve{\Pi}_x(t) = \frac{1}{m_\pi(x)} \int_0^{\Pi^{-1}(t)} f(x \mid \tau)\pi(\tau)d\tau, \quad t \in (0,1).$$

This is the value of the posterior cumulative distribution function at the prior t-quantile, i. e. $\breve{\Pi}_x(t) = \Pi^*(\Pi^{-1}(t))$. It is a cumulative distribution function (on the interval $(0,1)$) as well. It holds $\breve{\Pi}_x(\Pi(\theta)) = \Pi^*(\theta)$. It can be shown as follows.

Theorem 8.[37] Under the notation as above we have

(i) $\Pi \leq_{LR} \Pi^* \Leftrightarrow \breve{\Pi}_x$ is convex on the interval $(0,1)$ and
$\Pi \geq_{LR} \Pi^* \Leftrightarrow \breve{\Pi}_x$ is concave on the interval $(0,1)$;

(ii) $\Pi \leq_{st} \Pi^* \Leftrightarrow \breve{\Pi}_x(u) \leq u$ for any $u \in (0,1)$ and
$\Pi \geq_{st} \Pi^* \Leftrightarrow \breve{\Pi}_x(u) \geq u$ for any $u \in (0,1)$.

Example. Let us consider the Bayesian normal model with the mean as a parameter. Let the prior distribution be normal $N(\mu_\pi, \tau_\pi^2)$, i.e. with the cumulative distribution function $\Pi(\theta) = \Phi\left(\frac{\theta - \mu_\pi}{\tau_\pi}\right)$. The posterior cdf is then $\Pi^*(\theta) = \Phi\left(\frac{\theta - \mu_x}{\tau_x}\right)$, where μ_x and τ_x may be computed by well known formulae. Then

$$\breve{\Pi}_x(u) = \Pi^*(\Pi^{-1}(u)) = \Phi\left(\frac{\Pi^{-1}(u) - \mu_x}{\tau_x}\right).$$

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[35] J. Bartoszewicz, M. Skolimowska, op.cit.

[36] P. Błażej, *Preservation of classes of life distributions under weighting with a general weight function*, "Statistics and Probability Letters" 2008, vol. 78, pp. 3056–3061.

[37] P. Błażej, op.cit.

A quantile of a normal distribution is easy to compute with the quantile of the standard normal distribution $N(0,1)$ as $\Pi^{-1}(u) = \tau\Phi^{-1}(u) + \mu$. This implies

$$\breve{\Pi}_x(u) = \Phi\left(\frac{\tau_\pi}{\tau_x}\Phi^{-1}(u) - \frac{\mu_x - \mu_\pi}{\tau_x}\right).$$

Now

$$(\forall u \in (0,1))\, \breve{\Pi}_x(u) \le u \Leftrightarrow$$

$$(\forall u \in (0,1))\, \Phi^{-1}(u)\left(\frac{\tau_\pi}{\tau_x} - 1\right) \le \frac{\mu_x - \mu_\pi}{\tau_x} \Leftrightarrow$$

$$(\forall y \in \mathbf{R})\left(\frac{\tau_\pi}{\tau_x} - 1\right) y \le \frac{\mu_x - \mu_\pi}{\tau_x} \Leftrightarrow$$

$$(\forall y \in \mathbf{R})\left(\tau_\pi - \tau_x\right) y \le \mu_x - \mu_\pi,$$

what may be true only for $\tau_\pi = \tau_x$ but this does not hold. This means that in the Bayesian normal model we have not usual stochastic order between prior and posterior distributions (which is not surprising). Consequently, the LR ordering does not hold, either. ■

Let us consider another aspect of comparing distribution in respect of stochastic orders. Lehmann and Rojo[38] and also Frąszczak and Bartoszewicz[39] investigated pairs of distributions in regard to further or closer relative location of distributions each from or to other. This requires a precise definition which will be given below. Thus we ask whether the ordering for a pair of prior distributions $\Pi_1 \le_{LR} \Pi_2$ may imply that for posterior distributions satisfying $\Pi_1^* \le_{LR} \Pi_2^*$ the distribution Π_2^* is further to the right of Π_1^* than Π_2 is from Π_1. The precise definition of the expression "is further to the right" was given by Lehmann and Rojo.[40] To avoid developing more theory than actually needed, we give a precise but not the most intuitive form of the definition.

[38] E.I. Lehmann, J. Rojo, op.cit.
[39] M. Frąszczak, J. Bartoszewicz, op.cit.
[40] E.I. Lehmann, J. Rojo, op.cit. See also M. Frąszczak, J. Bartoszewicz, op.cit.

Definition 5. Let $F_1 \leq_{LR} G_1$ and $F_2 \leq_{LR} G_2$. We say that the distribution G_2 is further to the right of F_2 than G_1 is from F_1 if

$$G_1 F_1^{-1} \leq_{LR} G_2 F_2^{-1}.$$

Remarks. (i) The functions $G_i F_i^{-1}$, $i = 1,2$, are cumulative distribution functions. (ii) The third equivalent condition for the LR order implies that the condition $G_1 F_1^{-1} \leq_{LR} G_2 F_2^{-1}$ is equivalent to convexity of the function $G_2 F_2^{-1} F_1 G_1^{-1}$.

Moreover, this notion is related to a metric in the space of probability distributions. Lehmann and Rojo[41] define *consistency* of a metric with a fixed stochastic order. Details do not matter here, but this theory results in that the metric consistent with the LR order is

$$d(F,G) = \sup_x \left| \log \frac{g(x)}{f(x)} \right|.$$

We have also the following theorem.

Theorem 9.[42] Under the conditions of Definition 5, if the distribution G_2 is further to the right of F_2 than G_1 is from F_1 then $d(F_1, G_1) \leq d(F_2, G_2)$.

Certainly there is no equivalence, since the inequality for the distance does not imply the ordering of pairs of distributions.

Let us rewrite the above results for prior and posterior distributions.

Corollary. If $\Pi_1 \leq_{LR} \Pi_2$, what follows $\Pi_1^* \leq_{LR} \Pi_2^*$, then:

(i) Π_2^* is further to the right of Π_1^* than Π_2 is from Π_1 if and only if the function $\Pi_2^* (\Pi_1^*)^{-1} \Pi_1 \Pi_2^{-1}$ is convex;

(ii) if Π_2^* is further to the right of Π_1^* than Π_2 is from Π_1 then

$$\sup_{\theta \in \Theta} \left| \log \frac{\pi_2(\theta)}{\pi_1(\theta)} \right| \leq \sup_{\theta \in \Theta} \left| \log \left(\frac{\pi_2(\theta)}{\pi_1(\theta)} \cdot \frac{m_{\pi_1}(x)}{m_{\pi_2}(x)} \right) \right|.$$

Let us comment these results as follows: analysis of convexity of the composed function $\Pi_2^* (\Pi_1^*)^{-1} \Pi_1 \Pi_2^{-1}$ is involved even for simple Bayes models. Further, when considering the necessary condition from Theorem 9 we observe the influence of the factor $\frac{m_{\pi_1}(x)}{m_{\pi_2}(x)}$. In Bayesian analysis the value of the marginal

[41] E.I. Lehmann, J. Rojo, op.cit.

[42] Ibidem.

density $m_\pi(x)$ for current data x is used as an indicator of how much the prior agrees with the data. If it is close to 0, the correspondence is weak. Now, if for a given data point $x \in X$ we have $m_{\pi_1}(x)$ close to 0 and $m_{\pi_2}(x)$ moderate or large (or reversely), then the value of this factor make the argument of the logarithmic function close to 0 (or large). Then the necessary condition that Π_2^* is further to the right of Π_1^* than Π_2 is from Π_1 is satisfied. This is intuitively right, because very large or very close to 0 value of $\frac{m_{\pi_1}(x)}{m_{\pi_2}(x)}$ for a given x means that the values of $m_{\pi_i}(x)$, $i = 1.2$ are not close each to other, what means further that one of the priors Π_i much better corresponds to the data that the other one. And this should result in a bigger distance of posterior distributions (given the data x) than of the prior ones.

However, practical computations for fixed statistical models may be troublesome. The distance defined as above may be infinite for pairs of distributions in many useful statistical models, in particular for the families with monotone likelihood ratio, since the ratio may be unbounded. For example in one-parameter exponential families of the form

$$f(t \mid \lambda) = h(t)e^{c(\lambda)T(t) - b(\lambda)}$$

we obtain the logarithm of the likelihood ratio in the form

$$\log \frac{f(t \mid \lambda_1)}{f(t \mid \lambda_2)} = T(t)(c(\lambda_1) - c(\lambda_2)),$$

which may be easily growing to infinity in t.

6. Stochastic orderings for predictive distributions

Statistical prediction consists in predicting an unknown (unobservable, future) value of a random variable Y on the basis of a random sample $Z_n = (X_1, \ldots, X_n)$. It can be made with a statistic $\hat{Y}_n = \hat{Y}(Z_n)$ which minimises the expected loss (the expected prediction error) $\Delta = EL(Y, \hat{Y}_n)$ and it is well known that for the quadratic loss function the optimum predictor is $E(Y \mid Z_n = z_n)$. Also predictive confidence intervals may be constructed. When using the Bayesian statistical model we can construct predictive distribution which in prediction is analogous to the posterior distribution in estimation.[43]

Definition 6. Let us assume that the predicted variable Y has a conditional density $g(y|\theta,z_n)$. The posterior distribution of θ is denoted by $\pi(\theta|z_n)$. The predictive distribution of Y under the data z_n and the prior distribution π is the conditional distribution with the density function

$$p_G^\pi(y|z_n)=\int_\Theta g(y|\theta,z_n)\pi(\theta|z_n)d\theta.$$

It has the structure of a mixed distribution and is a conditional marginal distribution of Y under z_n. For the simple case of $Y=X_{n+1}$ with X_i, $i=1,2,\dots,n+1$, conditionally independent under θ we obtain $p(x_{n+1}|z_n)=\int_\Theta f(x_{n+1}|\theta)\pi(\theta|z_n)d\theta$, i.e. the marginal distribution of a single observation in case when the posterior distribution takes the role of the prior.

For the predictive distributions we make use of the results on the ordering of marginal distributions (mixtures) and of posterior distributions. We obtain the properties as follows:

Theorem 10. (1) The usual stochastic order of distributions of the predicted variable (conditional in θ) may be transferred to predictive distributions for any given data z_n, that is if

$$(\forall\theta\in\Theta)\,F(\cdot|\theta,z_n)\le_{st} G(\cdot|\theta,z_n)$$

then

$$F_p^\pi(\cdot|z_n)\le_{st} G_p^\pi(\cdot|z_n),$$

where $F(\cdot|\theta,z_n)$ means the cumulative distribution function corresponding to the density $f(\cdot|\theta,z_n)$); for $G(\cdot|\theta,z_n)$ – analogously; and $F_p^\pi(\cdot|z_n)$ denotes the predictive cdf corresponding to the predictive density

$$p_F^\pi(y|z_n)=\int_\Theta f(y|\theta,z_n)\pi(\theta|z_n)d\theta,$$

$G_p^\pi(\cdot|z_n)$ – analogously.

(2) If for distributions of the predicted variable Y we have

$$(\forall z_n,\theta\le\theta',\theta,\theta'\in\Theta)\,F(\cdot|\theta,z_n)\le_{LR} F(\cdot|\theta',z_n)$$

[43] See C. Robert, op.cit.

and if $\Pi_1 \leq_{LR} \Pi_2$, then for the predictive distributions

$$F_p^{\pi_1}(\cdot \mid z_n) \leq_{LR} F_p^{\pi_2}(\cdot \mid z_n).$$

The conclusion (1) is a natural consequence of the form of predictive distributions as mixtures and of Theorems 1 and 2. The conclusion (2) for predictive distributions requires the assumption on the LR ordering, because the usual stochastic ordering may be transferred to posterior distribution only under LR ordering.

Stochastic orders for posterior and predictive distributions imply comparison properties for estimators and predictors. This problem is addressed by Nowak[44] and Bartoszewicz and Nowak.[45] In the latter paper, the coincidence between posterior distributions and weighted prior distributions with likelihood functions as weight functions seems to be explicitly noticed for the first time in the literature.

7. Increasing convex (stop-loss) order

Finally we turn to another stochastic ordering which has important applications to insurance, i. e. the increasing convex order known also as the stop-loss order.

Definition 7.[46] Let X and Y be random variables on a fixed probability space with cumulative distribution functions F and G, respectively. The random variable X is said to be less than Y with respect to the increasing $t \in \mathbf{R}$ convex order (we write $X \leq_{icx} Y$), if $Ef(X) \leq Ef(Y)$ for any increasing convex function *f*.

If the assumption on the monotonicity is relaxed, then we deal with the convex order, which we write as $X \leq_{cx} Y$. In particular, $X \leq_{cx} Y \Leftarrow X \leq_{icx} Y$ and $EX = EY$.

Remark 1. In insurance[47] mathematics the increasing convex order is called the stop-loss order: $X \leq_{SL} Y$, because $X \leq_{icx} Y \iff E(X-t)_+ \leq E(Y-t)_+$ for all.

44 P. Nowak, *Stochastic Ordering of Estimators* (in Polish), Ph.D. Dissertation, Institute of Mathematics, University of Wrocław 2012.

45 J. Bartoszewicz, P. Nowak, *Monotonicity of Bayes estimators*, "Applicationes Mathematicae" 2013, vol. 40, pp. 393–404.

46 M. Shaked, J.G. Shanthkumar, op.cit.

47 A. Müller, D. Stoyan, op.cit.

The function $\phi_X(t) = E(X-t)_+ = \int_t^{+\infty}(1-F_X(z))dz$ is called the integrated survival function or the stop-loss transform. This formula describes the optimum net stop-loss insurance premium in reinsurance contracts.

Remark 2. In terms of the integrated survival function we can also characterise the usual stochastic order, since $X \leq_{st} Y \iff \varphi_Y(t)-\varphi_X(t)$ is a decreasing function. Of course, $X \leq_{SL} Y \iff \varphi_Y(t)-\varphi_X(t) \geq 0$ for all $t \in \mathbf{R}$. We can easily see that $X \leq_{st} Y$ implies $\varphi_Y(t)-\varphi_X(t) \geq 0$, so under $X \leq_{st} Y$ we have $X \leq_{SL} Y$ as well.

Properties.[48]

(1) If $X \leq_{icx} Y$ and Z is a random variable independent of X and Y, then $X+Z \leq_{icx} Y+Z$.

(2) Let X, Y and T be random variables such that the conditional distributions satisfy the following relation:

$$(\forall \theta \in \Theta) X \mid T = \theta \leq_{SL} Y \mid T = \theta$$

(this means that the definition of the stop-loss order is satisfied by f corresponding conditional distributions). Then $X \leq_{SL} Y$, which means that the stop-loss order is closed under mixtures and in Bayesian terms it may be transferred to marginal distributions of data. This is analogous to Theorem 1 and further similar properties.

(3) We have also a property analogous to Theorem 2: let us consider the family of distributions $\{F(\cdot \mid \theta), \theta \in \Theta\}$. Let $X(\theta)$ be a random variable with the distribution function $F(\cdot \mid \theta)$. For random variables T_i, $i=1,2$, sharing their support included in Θ and with a distribution functions Π_i, $i=1,2$, let $Y_i = X(T_i)$ denote random variables with the distribution functions H_i defined by

$$H_i(x) = \int_\Theta F(X \mid \theta) d\Pi_i(\theta), \quad x \in \mathbf{R}$$

If $X(\theta) \leq_{SL} x(\theta')$ for all $\theta, \theta' \in \Theta$ such that $\theta \leq \theta'$ and if $T_1 \leq_{SL} T_2$, then $Y_1 \leq_{SL} Y_2$.

48 A. Müller, D. Stoyan, op.cit.; M. Shaked, J.G. Shanthikumar, op.cit.

(4) We have not got a property of transferring the stop-loss order from sampling or prior distributions onto posterior distributions without additional assumptions. Recall that if $X \leq_{LR} Y$ then $X \leq_{st} Y$, which implies $X \leq_{SL} Y$.

This allows us to make use of Theorems 5 and 6 assuming that the variables of interest ordered with respect to the stop-loss order are also ordered with respect to the likelihood ratio order. Or easier, if we start from the likelihood ratio order which is transferred to posterior, marginal and predictive distributions, we arrive at the transferring of the stop-loss order to resulting distributions.

However, there exist pairs of random variables which are ordered with respect to the stop-loss order and are not with respect to the usual stochastic one and consequently with respect to the likelihood ratio order.

Definition 8. A random variable X is said to be less dangerous than a variable Y, if there exists a point $t_0 \in \mathbf{R}$ such that $(\forall t < t_0) F_X(t) \leq F_Y(t)$ and $(\forall t \geq t_0) F_X(t) \geq F_Y(t)$ with $EX \leq EY$.

Theorem 11.[49] Let X and Y be random variables on a fixed probability space with cumulative distribution functions F and G, respectively. If X is less dangerous than Y, then $X \leq_{SL} Y$.

The assumption on the intersection of cumulative distribution functions contradicts the definition of the usual stochastic order and consequently the likelihood ratio order, although the random variables under consideration satisfy the stop-loss order.

Let us show some examples on how inequalities for parameters of distributions correspond to stochastic orders and on implications of stochastic orders.

Examples.

(1) It is known[50] that if $X \sim N(\theta_X, \sigma_X^2)$ and $Y \sim N(\theta_Y, \sigma_Y^2)$, then the inequality $\theta_X \leq \theta_Y$ with $\sigma_X = \sigma_Y$ implies $X \leq_{LR} Y$. If we allow $\sigma_X \leq \sigma_Y$, then $X \leq_{SL} Y$, but for $\sigma_X \neq \sigma_Y$ the relation $X \leq_{LR} Y$ does not hold.

Now let Z have a normal distribution $N(\theta, \sigma^2)$ with the normal prior distribution $N(\mu, \tau)$ for the mean. Then the posterior distribution has the form

$N(\mu_Z, \sigma_Z^2)$, where $\mu_Z = \dfrac{\sigma^2 \mu + \tau^2 z}{\sigma^2 + \tau^2}$ and $\sigma_Z = \dfrac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}$. Let us consider two

normal priors Π_i in the form $N(\mu_i, \tau_i^2)$, $i = 1,2$, $\mu_1 \leq \mu_2$. Then the ordering of means is preserved for posterior means under $\tau_1 = \tau_2$, what implies the

49 A. Müller, D. Stoyan, op.cit.

50 Ibidem.

likelihood ratio order and consequently the stop-loss order of posterior distributions. If we allow $\tau_1 < \tau_2$, then the priors Π_1 i Π_2 are ordered with respect to the stop-loss order, but the corresponding posteriors are ordered only for $z > \mu$.

(2) Let $X \sim Poiss(\theta)$; assume the conjugate prior distribution, that is $Gamma(\beta,\alpha)$ with the density function $\pi(\theta) = \frac{\alpha^\beta}{\Gamma(\beta)}\theta^{\beta-1}e^{-\alpha\theta}$, $\theta > 0$. The posterior distribution $\Pi(\cdot\,|\,x)$ is $Gamma(\beta + x, \alpha + 1)$.
Let us consider two gamma priors Π_i in the form $Gamma(\beta_i,\alpha_i)$, $i = 1,2$. Then for $\beta_1 \le \beta_2 \;\wedge\; \alpha_1 \ge \alpha_2$ we obtain $\Pi_1 \le_{LR} \Pi_2$,[51] hence $\Pi_1 \le_{SL} \Pi_2$. Since we have also $\beta_1 + x \le \beta_2 + x \;\wedge\; \alpha_1 + 1 \ge \alpha_2 + 1$, so for the posterior distributions we have $\Pi_1(\cdot\,|\,x) \le_{LR} \Pi_2(\cdot\,|\,x)$, hence $\Pi_1(\cdot\,|\,x) \le_{SL} \Pi_2(\cdot\,|\,x)$. Moreover, the ordering $\Pi_1 \le_{SL} \Pi_2$ is implied by the inequalities $\beta_1 \ge \beta_2 \;\wedge\; \frac{\beta_1}{\alpha_1} \le \frac{\beta_2}{\alpha_2}$ as well.[52]
Then we have $\beta_1 + x \ge \beta_2 + x$, but $\frac{\beta_1 + x}{\alpha_1 + 1} \le \frac{\beta_2 + x}{\alpha_2 + 1}$ only for
$x \ge \frac{\beta_1 - \beta_2 - (\beta_2\alpha_1 - \beta_1\alpha_2)}{\alpha_1 - \alpha_2}$.

(3) Let $X \sim Ex(\theta)$, which means that $f(x) = \theta e^{-\theta x}$, $x > 0$. Let us assume the conjugate prior $Gamma(\beta,\alpha)$. The posterior distribution $\Pi(\cdot\,|\,x)$ is *Gam-ma*$(\beta + 1, \alpha + x)$.
Let us consider two gamma priors Π_i in the form $Gamma(\beta_i,\alpha_i)$, $i = 1,2$. Again for $\beta_1 \le \beta_2 \;\wedge\; \alpha_1 \ge \alpha_2$ we obtain $\Pi_1 \le_{LR} \Pi_2$, hence $\Pi_1 \le_{SL} \Pi_2$. But this implies $\beta_1 + 1 \le \beta_2 + 1 \;\wedge\; \alpha_1 + x \ge \alpha_2 + x$, so for the posterior distributions it holds $\Pi_1(\cdot\,|\,x) \le_{LR} \Pi_2(\cdot\,|\,x)$, hence $\Pi_1(\cdot\,|\,x) \le_{SL} \Pi_2(\cdot\,|\,x)$. As before, the ordering $\Pi_1 \le_{SL} \Pi_2$ is also implied by the inequalities $\beta_1 \ge \beta_2 \;\wedge\; \frac{\beta_1}{\alpha_1} \le \frac{\beta_2}{\alpha_2}$. Then we obtain $\beta_1 + 1 \ge \beta_2 + 1$, but $\frac{\beta_1 + 1}{\alpha_1 + x} \le \frac{\beta_2 + 1}{\alpha_2 + x}$ only for $x \le \frac{\beta_2\alpha_1 - \beta_1\alpha_2 + \alpha_1 - \alpha_2}{\beta_1 - \beta_2}$.

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[51] Ibidem.
[52] Ibidem.

As we can see, the assumption of the likelihood ratio order is essential for transferring the usual stochatic and stop-loss orders from sampling or prior distributions to posterior distributions.

We recall that the stop-loss order is important because of the significance of the stop-loss transform for computing a premium, including the optimum reinsurance contract. In particular, it is known that stop-loss-larger claims yield larger ruin probabilities.[53] It is a straightforward consequence of the stop-loss order for risks when the expectations are constant that the variance and standard deviation premium principles yield increasing premiums.[54] Moreover, the exponential premium principle (and, in general, the zero utility premium) results in a premium increasing with respect to the stop-loss order of risk.[55] As a particular case of it the Bayes premium with respect to the LINEX loss[56] can be seen.

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[53] R. Kaas et al., op.cit.

[54] W.R. Heilmann, K.J. Schröter, op.cit.

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Porządki stochastyczne w aspekcie bayesowskim

Streszczenie

Praca stanowi przegląd problematyki porządków stochastycznych w aspekcie bayesowskim, to znaczy stochastycznego uporządkowania rozkładów *a posteriori*, brzegowych rozkładów obserwacji i rozkładów predyktywnych przy założeniach porządkowych dla rozkładów obserwacji i rozkładów *a priori*. Podano komentarze na temat znaczenia dla teorii ryzyka i zastosowań aktuarialnych.

Słowa kluczowe: zwykły porządek stochastyczny, porządek dyspersyjny, porządek ilorazowy, porządek rosnący wypukły (*stop-loss*), rozkłady ważone, rozkłady *a priori*, rozkłady *a posteriori*, rozkłady predyktywne, teoria ryzyka