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A method of calculating exact ruin probabilities in discrete time models

Summary

The paper presents an application of an integral operator generated by the discrete time risk process to determining the exact formulae for ruin probabilities. The methodology is based on finding a fixed point of the operator and verifying whether it is identically equal to the probability of ruin. The exact ruin probabilities are derived for an absolutely continuous as well as for a discrete amount distribution of claims. Numerical examples are also given.

Keywords: discrete time risk models, ruin probabilities, integral operator generated by a risk process, theory of fixed points, Solvency II

1. Introduction

Each insurer should examine its financial situation and report on it to its supervisor. The reports are usually sent at the end of each fixed time period. Thus, the discrete time setup seems to be relevant in the area of real-life applications.[2] The importance of the discrete time models might increase if one uses them as internal models according to the forthcoming EU directive Solvency II.

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2 Cf. H. Jasiulewicz, *Dyskretny proces ryzyka z uwzględnieniem reasekuracji i losowej stopy procentowej [Discrete risk process with reinsurance and random interest rate]*, „Roczniki Kolegium Analiz Ekonomicznych", Warsaw School of Economics, Warsaw 2013, vol. 31, pp. 11–26; S. Cheng, H.U. Gerber, E.S.W. Shiu, *Discounted probabilities and ruin theory in the compound binomial model*, "Insurance: Mathematics and Economics" 2000, vol. 26, pp. 239–250; L. Gajek, M. Rudź, *Sharp approximations of ruin probabilities in the discrete time models*, "Scandinavian Actuarial Journal" 2013, pp. 352–382.

Many approximations of ruin probabilities have been proposed.[3] To investigate the accuracy of an approximation, it is necessary to know the exact values of ruin probabilities. They provide a good benchmark for numerical studies. The problem of the exact ruin probabilities was investigated in continuous time models.[4] However, the actuarial literature related to this problem in the discrete time case is rather scarce.

2. A discrete time risk model

All stochastic objects considered in the paper are assumed to be defined on a probability space $(\Omega, \Im, P)$. Let N denote the set of all positive integers and R, the real line. Set $N^0 = N \cup \{0\}$, $R_+ = (0, \infty)$, $R_+^0 = [0, \infty)$, and $\overline{R}_+ = (0, \infty]$.

In this paper, we investigate the following discrete time risk model. Let a non-negative random variable X_i denote the aggregated sum of the claims in the i th time period; a positive real γ, the amount of aggregated premiums received each period; and a non-negative real u, the insurer's surplus at 0. We assume that $X_1, X_2, \ldots$ are i.i.d. random variables with a common distribution function F. Let $\{S(n)\}_{n \in N^0}$ denote the insurer's surplus process defined by

[3] To list only a few recent monographs and papers: S. Asmussen, H. Albrecher, *Ruin probabilities*, 2nd ed., World Scientific, Singapore 2010; P. Čížek, W. Härdle, R. Weron, *Statistical tools for finance and insurance*, Springer-Verlag, Berlin Heidelberg 2005; J. Grandell, *Simple approximations of ruin probabilities*, "Insurance: Mathematics and Economics" 2000, vol. 26, pp. 157–173; T. Rolski, H. Schmidli, V. Schmidt, J. Teugels, *Stochastic processes for insurance and finance*, Wiley, New York 1999.

[4] To list only a few recent monographs and papers: B. Chan, H.U. Gerber, E.S.W. Shiu, *Discussion of papers already published – "On a classical risk model with a constant dividend barrier" by X. Zhou*, "North American Actuarial Journal" 2006, vol. 10, 2, pp. 133–139; J.M.A. Garcia, *Explicit solutions for survival probabilities in the classical risk model*, "ASTIN Bulletin" 2005, vol. 35, 1, pp. 113–130; S.A. Klugman, H.H. Panjer, G.E. Willmot, *Loss models. From data to decisions*, Wiley, New York 1998; N.L. Bowers, H.U. Gerber, J.C. Hickmann, D.A. Jones, C.J. Nesbitt, *Actuarial Mathematics*, 2nd ed., The Society of Actuaries, Schaumburg, IL 1997; J. Babier, B. Chan, *Approximations to ruin probability by di-atomic or di-exponential claims*, "ASTIN Bulletin" 1992, vol. 22, 2, pp. 235–246; B. Chan, *Ruin probability for translated combination of exponential claims*, "ASTIN Bulletin" 1990, vol. 20, pp. 113–114; F. Dufresne, H.U. Gerber, *The probability and severity of ruin for combinations of exponential claim amount distributions and their translations*, "Insurance: Mathematics and Economics" 1988, vol. 7, pp. 75–80; F. Dufresne, H.U. Gerber, *Three methods to calculate the probability of ruin*, "ASTIN Bulletin" 1989, vol. 19, pp. 71–90; F. Dufresne, H.U. Gerber, *Rational ruin problems – a note for the teacher*, "Insurance: Mathematics and Economics" 1991, vol. 10, pp. 21–29.

$$S(0) = u$$

and

$$S(n) = u + \gamma n - \sum_{i=1}^{n} X_i, \;\; n \in N.$$

The above model can be found e.g. in the papers by Rolski et al.,[5] Klugman et al.,[6] Gajek[7] or Gajek and Rudź[8] among many others.

Throughout the paper we will use the convention that $\inf \varnothing$ means ∞. Then the moment of ruin is defined by

$$\tau = \tau(u) = \inf\left\{ n \in N : \;\; S(n) < 0 \right\}$$

and the infinite horizon probability of ruin by

$$\Psi(u) = P(\tau(u) < \infty).$$

Fix $\gamma \in R_+$. Let $M: \;\; R \to \overline{R_+}$ be defined by

$$M(r) = Ee^{-r(\gamma - X_1)}, \;\; r \in R.$$

The positive real solution r_0 of the following equation

$$M(r_0) = 1,$$

if it exists, is called adjustment coefficient. Write

$$R_0(u) = e^{-r_0 u}, \;\; u \geq 0$$

and

$$M_0 = \left\{ r \in R : \;\; M(r) < \infty \right\}.$$

The following result provides a sufficient condition for the existence of r_0.

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5 T. Rolski et al., op.cit.

6 S.A. Klugman et al., op.cit.

7 L. Gajek, *On the deficit distribution when ruin occurs-discrete time model*, "Insurance: Mathematics and Economics" 2005, vol. 36, pp. 13–24.

8 L. Gajek, M. Rudź, op.cit.

Lemma 1.[9] Assume that $EX_1 < \gamma$, $P(X_1 > \gamma) > 0$ and the set M_0 is open. Then there exists a unique adjustment coefficient $r_0 > 0$.

Under the assumptions of Lemma 1, one can show that

$$\Psi(u) = \frac{R_0(u)}{E[R_0(S(\tau)) \big| \tau < \infty]} , \tag{1}$$

which implies the following Cramér-Lundberg bound.[10]

$$\Psi(u) \le R_0(u)$$

Formula (1) leads to an applicable representation of ruin probabilities Ψ only in some special cases of F, including exponential and the two-point distributions. The next section concerns a more constructive method of calculating Ψ.

3. An operator-like approach

In this section we summarise[11] the relevant material on an integral operator generated by the risk process.

We will denote by $\Re$ the set of all non-increasing functions defined on R_+^0 and taking values in $[0, 1]$. Fix $\gamma \in R_+$ and F. A function $L: \Re \to \Re$ is said to be the integral operator generated by the risk process $\{S(n)\}_{n \in N^0}$ if

$$L\rho(u) = \int_0^{u+\gamma} \rho(u + \gamma - x)\, dF(x) + 1 - F(u + \gamma) , \ \rho \in \Re , \tag{2}$$

where the integral is considered over the interval $[0, u + \gamma]$. Gajek[12] proved, among other things, that the probability of ruin Ψ is a fixed point of L, i.e.

$$\Psi(u) = L\Psi(u) , \ u \ge 0 . \tag{3}$$

9 L. Gajek, op.cit., p. 15.
10 N.L. Bowers et al., op.cit.
11 After L. Gajek, op.cit. and L. Gajek, M. Rudź, op.cit.
12 L. Gajek, op.cit.

Formula (3) suggests an idea of deriving the exact ruin probabilities Ψ. One can fix the claim distribution F in (2) and solve then the integral equation (3). However, the operator L may have distinct fixed points.[13] A question arises: is the determined fixed point identically equal to Ψ ? The following proposition gives an answer.

Proposition 1. Let the assumptions of Lemma 1 hold. Assume that a function $\rho \in \Re$ is such that

$$\rho(u) = L\rho(u) \tag{4}$$

and

$$\rho(u) \leq R_0(u) \tag{5}$$

for every $u \geq 0$. Then $\Psi(u) = \rho(u)$, u≥0.
Proof: L. Gajek, M. Rudź, op.cit. ■

Thus, solving the integral equation (4) and proving that its solution is not greater than the Cramér-Lundberg bound, one may determine the explicit ruin probabilities Ψ. In the following section we will calculate[14] the exact formulae for ruin probabilities for some distributions of F. We consider a mixture of exponential distributions and the geometric distribution.

4. Main results

In this section we determine some formulae for the exact ruin probabilities. We apply the method described in Section 3. The first result concerns the case when F is a mixture of exponential distributions. Set $1(x \in A) = 1$ if $x \in A$ and 0, otherwise.

Theorem 1. Assume that X_1's density function is given by

$$f(x) = \sum_{i=0}^{n-1} A_i \beta_i e^{-\beta_i x} 1(x > 0),$$

[13] For details see L. Gajek, op.cit.

[14] After M. Rudź, *Wzory dokładne i przybliżone na prawdopodobieństwo ruiny w modelu dyskretnym [Exact and approximate formulae for the probability of ruin in a discrete time model]*, M.Sc. thesis, Technical University of Łódź, Faculty of Physics, Applied Mathematics and Computer Science, Institute of Mathematics, Łódź 2007.

where

$$n>1\,,\ 0<\beta_0<\beta_1<\ldots<\beta_{n-1}\ ,\ \sum_{i=0}^{n-1}A_i=1$$

and

$$A_i>0 \ \text{ for } i\in\{0,\ \ldots,\ n-1\}\ .\ \text{If } \gamma>\sum_{i=0}^{n-1}\frac{A_i}{\beta_i}$$

then

i) the following equations

$$e^{-r_k\gamma}\sum_{i=0}^{n-1}A_i\frac{\beta_i}{\beta_i-r_k}=1\ ,\ k\in\{0,\ \ldots,\ n-1\} \tag{6}$$

have exactly n real roots $r_0,\ \ldots,\ r_{n-1}$ such that $0<r_0<\beta_0<r_1<\beta_1<\ldots<r_{n-1}<\beta_{n-1}$;

ii) for every $u\geq 0$

$$\Psi(u)=\sum_{k=0}^{n-1}C_k e^{-r_k u}, \tag{7}$$

where

$$C_k=\prod_{\substack{i=0\\ i\neq k}}^{n-1}\left(\frac{r_i}{r_i-r_k}\right)\prod_{i=0}^{n-1}\left(\frac{\beta_i-r_k}{\beta_i}\right),\ k\in\{0,\ \ldots,\ n-1\}. \tag{8}$$

Proof: i) The function M in considered case is given by

$$M(r)=\int_0^\infty e^{-r(\gamma-x)}\sum_{i=0}^{n-1}A_i\beta_i e^{-\beta_i x}\,dx=\begin{cases} e^{-r\gamma}\sum_{i=0}^{n-1}A_i\dfrac{\beta_i}{\beta_i-r},\ r<\beta_0\\ \infty,\ r\geq\beta_0.\end{cases}$$

Thus, the set $M_0=(-\infty,\ \beta_0)$ is open. Since the support of the considered claim distribution is unbounded, the condition $P(X_1>\gamma)>0$ holds as well. Therefore, by Lemma 1, if

$$\gamma>EX_1=\sum_{i=0}^{n-1}\frac{A_i}{\beta_i}$$

then the adjustment coefficient r_0 exists with the understanding that $r_0\in(0,\ \beta_0)$. Let us define a function $\breve{M}$ by

$$\breve{M}(r) = e^{-r\gamma} \sum_{i=0}^{n-1} A_i \frac{\beta_i}{\beta_i - r}, \ r \geq 0, \ r \neq \beta_i, \ i \in \{0, \ \dots, \ n-1\}.$$

We see that $\lim_{r \to \beta_i^-} \breve{M}(r) = \infty$ and $\lim_{r \to \beta_i^+} \breve{M}(r) = -\infty$ for $i \in \{0, \ \dots, \ n-1\}$ and $\gamma < \infty$. The function $e^{r\gamma}\breve{M}(\cdot)$ is continuous and increasing in the interior of each interval $(\beta_0, \ \beta_1), \ \dots, \ (\beta_{n-2}, \ \beta_{n-1})$. Thus, in each of them its graph intersects exactly one time with the graph of the function $e^{r\gamma}$ and there exists exactly one solution, say $r_j, \ j \in \{1, \ \dots, \ n-1\}$, of the following equation

$$\breve{M}(r) = 1. \tag{9}$$

Summing up, under the assumption $\gamma > \sum_{i=0}^{n-1} \frac{A_i}{\beta_i}$, Equation (9) has exactly n real roots $r_0, \ \dots, \ r_{n-1}$ such that $0 < r_0 < \beta_0 < r_1 < \beta_1 < \dots < r_{n-1} < \beta_{n-1}$. A similar approach for the continuous time model can be found e.g. in the works of Bowers et al., Dufresne and Gerber and Otto.[15]

ii) Write

$$\rho(u) = \sum_{k=0}^{n-1} C_k e^{-r_k u}, \ u \geq 0, \tag{10}$$

where the coefficients C_k are given by (8).

We are now in a position to show that ρ given by (10) and (8) is a fixed point of the operator *L*. In order to prove this, we will use some computational techniques presented in the continuous time case by Dufresne and Gerber[16] and Chan.[17]

Note that $\rho \in \Re$ and recall that $\breve{M}(r_k) = 1$ for $k \in \{0, \ \dots, \ n-1\}$. Therefore,

$$L\rho(u) = \int_0^{u+\gamma} \rho(u+\gamma-x)\,dF(x) + 1 - F(u+\gamma)$$
$$= \int_0^{u+\gamma} \left(\sum_{k=0}^{n-1} C_k e^{-r_k(u+\gamma-x)} \sum_{i=0}^{n-1} A_i \beta_i e^{-\beta_i x} \right) dx + \sum_{i=0}^{n-1} A_i e^{-\beta_i(u+\gamma)}$$

[15] N.L. Bowers, op.cit.; F. Dufresne, H.U. Gerber, *Three methods…*, op.cit.; W. Otto, *Ubezpieczenia majątkowe. Część I. Teoria ryzyka [Non-life insurance. Part I. Risk theory]*, Wydawnictwa Naukowo-Techniczne, Warsaw 2004.

[16] F. Dufresne, H.U. Gerber, *Three methods…*, op.cit.

[17] B. Chan, op.cit.

$$= \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\frac{A_i\beta_i C_k}{\beta_i - r_k}e^{-r_k(u+\gamma)} - \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\frac{A_i\beta_i C_k}{\beta_i - r_k}e^{-\beta_i(u+\gamma)} + \sum_{i=0}^{n-1}A_i e^{-\beta_i(u+\gamma)}$$

$$= \sum_{k=0}^{n-1}\left(C_k e^{-r_k u}e^{-r_k\gamma}\sum_{i=0}^{n-1}\frac{A_i\beta_i}{\beta_i - r_k}\right) - \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\frac{A_i\beta_i C_k}{\beta_i - r_k}e^{-\beta_i(u+\gamma)} + \sum_{i=0}^{n-1}A_i e^{-\beta_i(u+\gamma)}$$

$$= \rho(u) - \sum_{i=0}^{n-1}\left(A_i e^{-\beta_i(u+\gamma)}\sum_{k=0}^{n-1}\frac{\beta_i C_k}{\beta_i - r_k}\right) + \sum_{i=0}^{n-1}A_i e^{-\beta_i(u+\gamma)},\ u \ge 0.$$

To prove (4), we have to show that

$$\sum_{i=0}^{n-1}\left(A_i e^{-\beta_i(u+\gamma)}\sum_{k=0}^{n-1}\frac{\beta_i C_k}{\beta_i - r_k}\right) = \sum_{i=0}^{n-1}A_i e^{-\beta_i(u+\gamma)}.$$

This will be proved by comparing the coefficients corresponding to the terms $A_i e^{-\beta_i(u+\gamma)}$ for $i \in \{0,\ \dots,\ n-1\}$. It suffices to show that $C_0,\ \dots,\ C_{n-1}$ given by (8) are solutions of the following equations

$$\sum_{k=0}^{n-1} C_k \frac{\beta_i}{\beta_i - r_k} = 1,\ i \in \{0,\ \dots,\ n-1\}. \tag{11}$$

We proceed in the same manner as Chan.[18] Let us consider the following equality

$$\sum_{i=0}^{n-1}\frac{x}{x-r_i}C_i = 1 - \prod_{i=0}^{n-1}\frac{r_i(x-\beta_i)}{\beta_i(x-r_i)},\ x \ne r_i,\ i \in \{0,\ \dots,\ n-1\} \tag{12}$$

of the two rational functions of degree n taking the same values at $n+1$ points (i.e. the value 0 at $x=0$ and, following (11), the value 1 at $x \in \{\beta_0,\ \dots,\ \beta_{n-1}\}$) and having the same domain $R \setminus \{r_0,\ \dots,\ r_{n-1}\}$.

Fix $k \in \{0,\ \dots,\ n-1\}$. Multiplying (12) by $x - r_k$ we get

$$\sum_{\substack{i=0\\ i\ne k}}^{n-1}\frac{x(x-r_k)}{x-r_i}C_i + xC_k = x - r_k - \left(\prod_{i=0}^{n-1}\frac{x-\beta_i}{\beta_i}\right)\prod_{i=0}^{n-1}r_i\left(\prod_{\substack{i=0\\ i\ne k}}^{n-1}\frac{1}{x-r_i}\right).$$

Taking $x = r_k$ yields

[18] B. Chan, op.cit.

$$r_k C_k = -\left(\prod_{i=0}^{n-1} \frac{r_k - \beta_i}{\beta_i}\right)\prod_{i=0}^{n-1} r_i \left(\prod_{\substack{i=0 \\ i\neq k}}^{n-1} \frac{1}{r_k - r_i}\right).$$

Since $k \in \{0, \ldots, n-1\}$ is arbitrary and r_k is positive, we conclude that (8) holds as well.

Note that we have already proved that ρ given by (10) and (8) is a fixed point of L. By Proposition 1, it remains to prove that (5) holds for every $u \geq 0$. Indeed, since $r_j > r_0$ for $j \in \{1, \ldots, n-1\}$, $\beta_{n-1} - r_k > 0$ and $C_k > 0$ for $k \in \{0, \ldots, n-1\}$, we have

$$\begin{aligned}
\rho(u) &= \sum_{k=0}^{n-1} C_k e^{-r_k u} < R_0(u)\sum_{k=0}^{n-1} C_k = R_0(u)\sum_{k=0}^{n-1} C_k \left(\frac{\beta_{n-1} - r_k}{\beta_{n-1} - r_k}\right) \\
&= R_0(u)\sum_{k=0}^{n-1} C_k \left(\frac{\beta_{n-1}}{\beta_{n-1} - r_k} - \frac{r_k}{\beta_{n-1} - r_k}\right) \\
&= R_0(u)\left(\sum_{k=0}^{n-1} C_k \frac{\beta_{n-1}}{\beta_{n-1} - r_k} - \sum_{k=0}^{n-1} C_k \frac{r_k}{\beta_{n-1} - r_k}\right) \\
&= R_0(u)\left(1 - \sum_{k=0}^{n-1} C_k \frac{r_k}{\beta_{n-1} - r_k}\right) \leq R_0(u),\ u \geq 0.
\end{aligned}$$

By Proposition 1,

$$\Psi(u) = \rho(u)\ ,\ u \geq 0\ ,$$

which completes the proof. ■

We will now consider the geometric distribution under the assumptions that γ , $u \in N$.

Theorem 2. Assume that X_1's probability function is given by

$$P(X_1 = n) = q^{n-1} p,\ n \in N\ ,$$

where $p \in (0,\ 1)$ and $q = 1 - p$. If a positive integer γ satisfies the inequality $\gamma > \dfrac{1}{p}$ then

$$\Psi(u) = e^{-r_0(u+\gamma)},\ u \in N\ ,$$

where $r_0 \in (0,\ \ln\frac{1}{q})$ is the adjustment coefficient.

Proof: The function M in the considered case is given by

$$M(r) = \sum_{n=1}^{\infty}\left(e^{-r(\gamma-n)}q^{n-1}p\right) = e^{-r\gamma}p\sum_{n=1}^{\infty}e^{r(n-1+1)}q^{n-1} = e^{-r\gamma}pe^{r}\sum_{n=1}^{\infty}\left(qe^{r}\right)^{n-1}$$
$$= \begin{cases} e^{-r\gamma}\dfrac{pe^{r}}{1-qe^{r}}, & r < \ln\dfrac{1}{q} \\ \infty, & r \ge \ln\dfrac{1}{q}. \end{cases}$$

Thus, the set $M_0 = (-\infty,\ \ln\frac{1}{q})$ is open. Since the support of the considered claim distribution is unbounded, the condition $P(X_1 > \gamma) > 0$ holds as well. Therefore, by Lemma 1, if $\gamma > EX_1 = \frac{1}{p}$ then the adjustment coefficient r_0 exists with the understanding that $r_0 \in (0,\ \ln\frac{1}{q})$.

Set

$$\rho(u) = e^{-r_0(u+\gamma)},\ u \in N. \tag{13}$$

We will prove that ρ given by (13) is a fixed point of the operator L.

Note that $\rho \in \Re$ and recall that $M(r_0) = 1$. Therefore,

$$L\rho(u) = \sum_{n=0}^{u+\gamma}\rho(u+\gamma-n)P(X_1 = n) + 1 - F(u+\gamma) = \sum_{n=1}^{u+\gamma}e^{-r_0(u+\gamma-n+\gamma)}q^{n-1}p + q^{u+\gamma}$$
$$= e^{-r_0(u+\gamma)}e^{-r_0\gamma}pe^{r_0}\sum_{n=1}^{u+\gamma}\left(e^{r_0}q\right)^{n-1} + q^{u+\gamma} = e^{-r_0(u+\gamma)}e^{-r_0\gamma}\frac{pe^{r_0}}{1-qe^{r_0}}\left(1-\left(qe^{r_0}\right)^{u+\gamma}\right) + q^{u+\gamma}$$
$$= e^{-r_0(u+\gamma)} = \rho(u),\ u \in N.$$

Since $\gamma > 0$, we have $\rho(u) = e^{-r_0(u+\gamma)} < R_0(u),\ u \in N$. By Proposition 1,

$$\Psi(u) = \rho(u),\ u \in N,$$

which completes the proof. ■

5. Numerical examples

On account of Lemma 1, we assume that $\gamma > EX_1$. In practically-oriented problems $\gamma = (1+\theta)EX_1$, where $\theta > 0$ is the relative security loading.[19]

Since the exact ruin probabilities Ψ were derived in Section 4, numerical computations are possible as well. We will present now the results of a simulation study. The computations were carried out by the author. The bisection method was used.

Example 1. Let us consider a mixture of five exponential distributions with scale parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (2, 4, 5, 6, 8)$ and weights $(A_0, A_1, A_2, A_3, A_4) = (0.2, 0.2, 0.2, 0.2, 0.2)$ respectively. Set $\gamma = 0.275 > 0.248 \approx EX_1$. It means that the relative security loading $\theta \approx 0.11$.

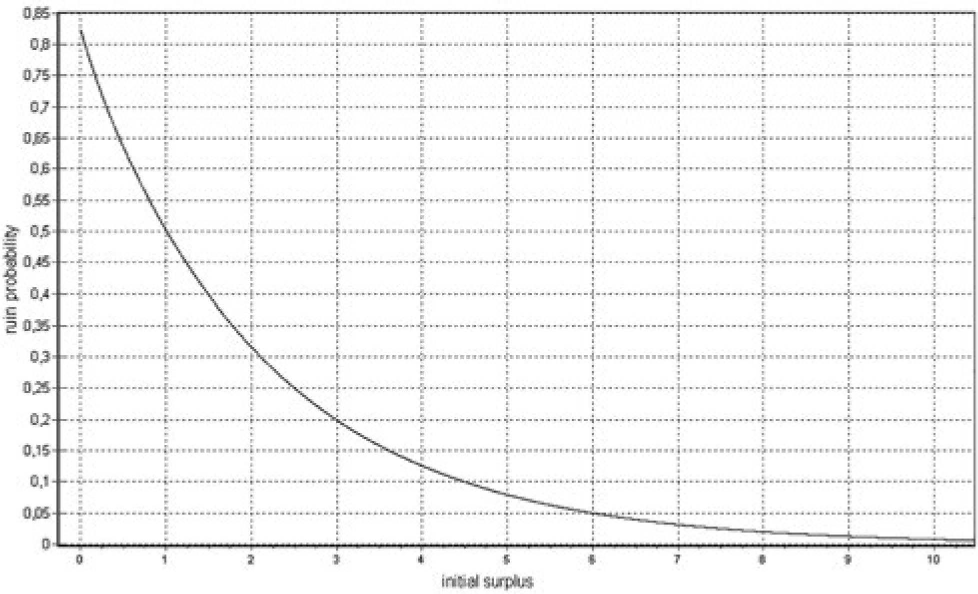

Figure 1. The exact probability of ruin $\Psi(u)$, $u \in [0, 10.4)$ for a mixture of five exponential distributions with scale parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) =$ $= (2, 4, 5, 6, 8)$ and weights $(A_0, A_1, A_2, A_3, A_4) = (0.2, 0.2, 0.2, 0.2, 0.2)$ respectively and the relative security loading $\theta \approx 0.11$

Source: own computations.

[19] Cf. N.L. Bowers et al., op.cit.

Figure 1 illustrates the exact probability of ruin $\Psi(u)$, $u \in [0, 10.4)$ computed numerically applying Formulae (6)–(8). Since θ is relatively small, the probability of ruin is relatively high, especially for $u < 2\gamma$, i.e. for the values of u that are usually important in real-life situations.

The following example concerns the impact of θ on Ψ.

Example 2. Let us consider a mixture of five exponential distributions with scale parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (2, 4, 5, 6, 8)$ and weights $(A_0, A_1, A_2$ $A_3, A_4) = (0.2, 0.1, 0.1, 0.2, 0.4)$ respectively. Set $\gamma = 0.3 > 0.228 \approx EX_1$. It means that the relative security loading $\theta \approx 0.31$.

The graph of Ψ is presented in Figure 2. In Figure 3 the same probability of ruin is compared together with the one corresponding to $\gamma = 0.35$ (i.e. $\theta \approx 0.53$). We see that an increase in the amount of premiums reduces the probability of ruin. Obviously, the level of θ should not be too high. Otherwise, potential policyholders may not be interested in taking out an insurance policy.

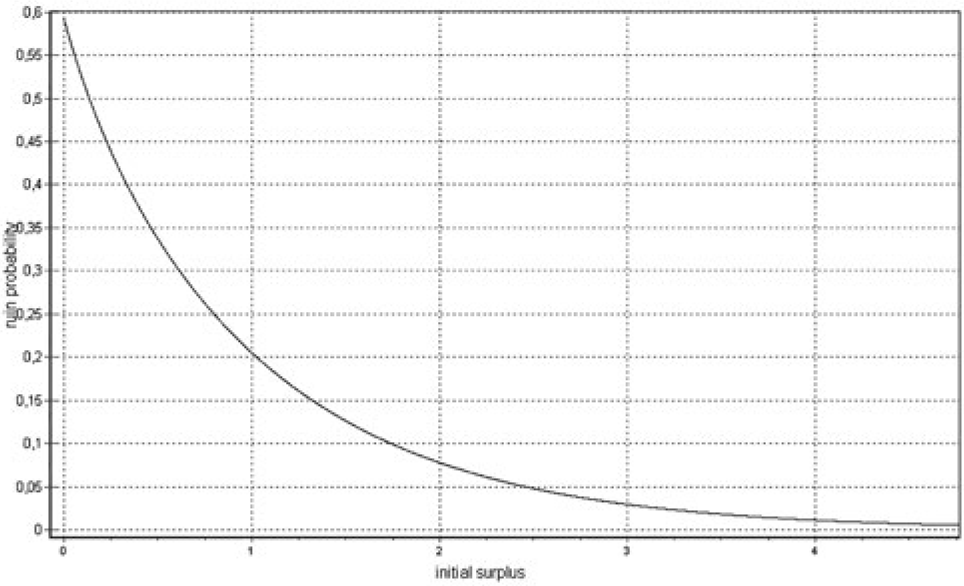

Figure 2. The exact probability of ruin $\Psi(u)$, $u \in [0, 5)$ for a mixture of five exponential distributions with scale parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (2, 4, 5, 6, 8)$, weights $(A_0, A_1, A_2, A_3, A_4) = (0.2, 0.1, 0.1, 0.2, 0.4)$ respectively and the relative security loading $\theta \approx 0.31$

Source: own computations.

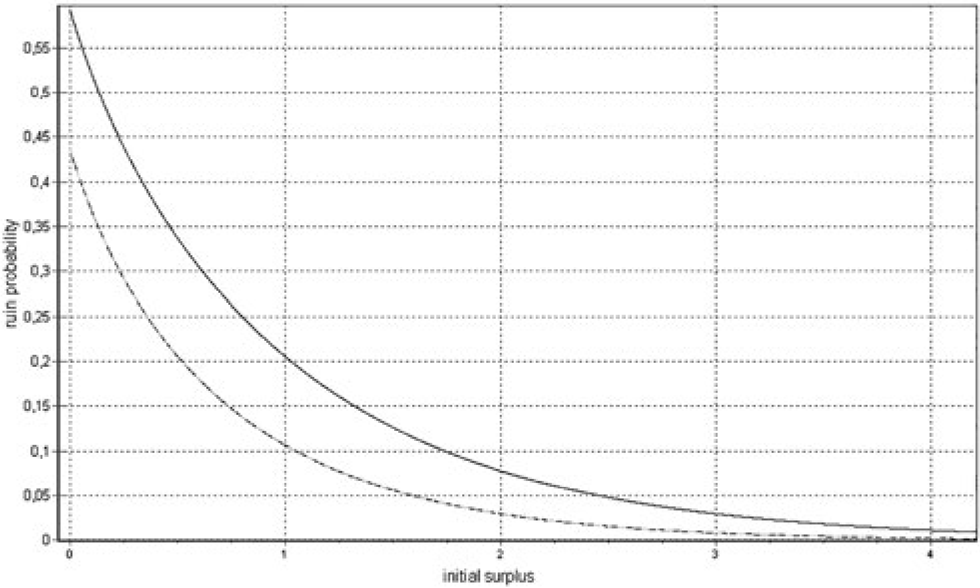

Figure 3. A mixture of five exponential distributions with scale parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = (2, 4, 5, 6, 8)$ and weights $(A_0, A_1, A_2, A_3, A_4) = (0.2, 0.1, 0.1, 0.2, 0.4)$ respectively. The exact ruin probabilities corresponding to $\theta \approx 0.31$ (the solid line) and $\theta \approx 0.53$ (the dashed line)

Source: own computations.

6. Conclusion

In this paper we investigated a method of determining the exact formulae for ruin probabilities in the discrete time framework. The method involves the integral operator L, defined by (2), which has a fixed point at the probability of ruin $\Psi(u)$. We conclude that the operator-like approach leads to an effective way of finding $\Psi(u)$. In particular, we derived it for mixtures of exponential distributions and the geometric amount distribution of claims.

A knowledge of the exact ruin probabilities enables one to investigate the accuracies of Ψ 's approximations. The problem might find an application also on account of the forthcoming EU directive Solvency II.

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Metoda wyznaczania dokładnych prawdopodobieństw ruiny w modelach dyskretnych

Streszczenie

W niniejszej pracy zaprezentowano zastosowanie operatora całkowego generowanego przez proces ryzyka z czasem dyskretnym do wyznaczania dokładnych wzorów na prawdopodobieństwo ruiny. Metodologia jest oparta na znajdowaniu punktu stałego operatora i weryfikowaniu, czy jest on tożsamościowo równy prawdopodobieństwu ruiny. Dokładne wzory są wyprowadzone zarówno dla absolutnie ciągłego, jak i dla dyskretnego rozkładu wysokości szkód. Podane są również przykłady numeryczne.

Słowa kluczowe: modele ryzyka z czasem dyskretnym, prawdopodobieństwo ruiny, operator całkowy generowany przez proces ryzyka, teoria punktów stałych, *Solvency* II (Wypłacalność II)