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The demand-supply model with expectations. Complex economic dynamics

Abstract

We investigate the dynamics of the nonlinear demand-supply model with expectations. We investigate the impact of expectations on the dynamics of the price. We determine the equilibria and investigate their local asymptotic stability. The global behaviour of the market is analysed numerically. We present the bifurcation diagrams for each parameter and localize those values, for which the system indicates complex behaviour. We investigate how the dynamics of the model depends on the parameters. We present analytical results whenever it is possible and numerical simulations of the more interesting occurrences.

Keywords: perfect competition, expectations, equilibrium, bifurcation, deterministic chaos.

1. Introduction

In economic modelling, many examples of cobweb chaos have been demonstrated. Some of the them include (Brock, Hommes, 1997; Chiarella, 1988; Hommes, 1991, 1994; Jensen, Urban, 1984; Nusse, Hommes, 1990); Hommes (1991) applies the concept of adaptive expectations in a cobweb model with a single producer to investigate the occurrence of strange and chaotic behavior, Hommes (1994) and Jensen and Urban (1984) used linear demand functions with nonlinear supply equations. These findings indicate that the nonlinear cobweb model may explain various irregular fluctuations observed in real economic data. In this paper, the cobweb model with nonlinear demand and piecewise linear supply function will be investigated.

We determine the equilibria and investigate their local asymptotic stability. Either simple or complex dynamics can occur around an equilibrium. In addition to an asymptotically stable equilibrium, unstable fluctuations can occur. Violation of stability conditions lead to the flip bifurcation. The global behaviour of the economy is analysed numerically. We present the bifurcation diagrams and localize those values, for which the system indicates chaotic or complex behaviour. We present analytical results whenever it is possible and numerical simulations of the more interesting occurrences.

2. The linear model

The standard linear demand-supply model is one of the simplest economic models. The model describes the price behaviour in a single market. We write p_t for price, Q_t^d for the demands of goods and Q_t^s for the supply for goods, all at time t. The model is given by the following equations:

$$Q_t^d = \alpha - \beta p_t,\ \ \alpha, \beta > 0, \tag{1}$$

$$Q_t^s = -\gamma + \delta p_{t-1},\ \ \gamma, \delta > 0, \tag{2}$$

$$\Delta p_t = p_t - p_{t-1} = j(Q_t^d - Q_t^s),\ \ j > 0. \tag{3}$$

Standard market equilibrium condition about market clearance at every point of time is replaced now by the adjustment mechanism given by the equation (3). Substituting equations (1) and (2) into the equation (3) we get first-order linear difference equation which describes dynamics of the price on the considered market:

$$p_t = \frac{1}{1 + j\beta} \left(j(\gamma + \alpha) + (1 - j\delta) p_{t-1} \right). \tag{4}$$

Equation (4) has one fixed point $(p_t = p_{t-1} = \bar{p})$

$$\bar{p} = \frac{\alpha + \gamma}{\beta + \delta}$$

which is an equilibrium price for the linear model $(Q_t^d(\bar{p}) = Q_t^s(\bar{p}))$. It is important to specify the conditions which must be satisfied so that the equilibrium price is stable. It is sufficient to determine the values of the parameters for which the general solution of the homogeneous equation associated with (4) is always converging to zero.

Proposition 1. *Fixed point* $\bar{p} = \dfrac{\alpha + \gamma}{\beta + \delta}$ *of the difference equation (4) is globally asymptotically stable iff* $0 < \delta < \beta + \frac{2}{j}$.

In the linear model where both the demand and supply functions are linear only three types of price dynamics may occur. It is possible to observe: convergence to an equilibrium price, convergence to a period two cycle or unbounded exploding price oscillations. In reaction to this weakness of the linear model the nonlinear model is proposed in the remainder of this paper.

The equilibrium price of the linear model, $\bar{p}$ is used in the next section and plays a crucial role in the expectations formation. The simplicity of the linear model makes it an excellent candidate for studying the effect of expectations on the price dynamics and stability of the equilibrium price.

3. Expectations

Households are bounded rationally due to non-sufficient information and computational power to derive fully optimal decisions. As a substitute they use simple heuristics which have proven to be useful in the past. I assume that the households use a mix of extrapolative and reverting expectation formation rules to forecast national income. The main objective of this paper is to examine the impact of the expectations on the dynamics of the price. The main modification of the linear demand-supply model is that household's demand depends on the expected price.

The aggregate expectation about price in period t are formed at the end of period $t-1$ as a weighted average of extrapolative ($E^1_{t-1}[Y_t]$) and reverting ($E^2_{t-1}[Y_t]$) expectations. Expectations are formed with reference to the fixed point of (4) which is a long-run equilibrium for price in the linear model, denoted in what follows as

$$\bar{p} = \frac{\alpha+\gamma}{\beta+\delta}.$$

Extrapolative (trend following) expectations are formalized as:

$$E^1_{t-1}[p_t] = p_{t-1} + \mu_1 \left(p_{t-1} - \bar{p}\right), \ \mu_1 > 0.$$

Equilibrium reverting expectations are described as:

$$E^2_{t-1}[p_t] = p_{t-1} + \mu_2 \left(\bar{p} - p_{t-1}\right), \ 0 < \mu_2 < 1.$$

It is assumed that larger deviations of the price decrease the weight put on extrapolative expectations. Households believe that extreme economic conditions are not sustainable. Formally, a rule describing weight put on extrapolative expectations becomes:

$$w_t = \frac{1}{1 + \left(\omega \frac{p_{t-1}-\bar{p}}{\bar{p}}\right)^2}, \ \omega > 0$$

The equation describing the expectations of the price in period t becomes:

$$E_{t-1}[p_t] = w_t E^1_{t-1}[p_t] + (1-w_t) E^2_{t-1}[p_t], \ 0 < w_t \leqslant 1.$$

4. The demand-supply model with expectations

4.1. Equilibrium. Local stability

The proposed model of perfectly competitive market includes two new assumptions on the demand side. Demand depends on the expected price level $E_{t-1}[p_t]$ in the current period. The second assumption made in the proposed model is the introduction of the upper limit on the volume of supply, which is related to the maximum level of production that can be realized by entrepreneurs. Production capacity in the short term, may not be sufficient to meet the demand reported. The standard equilibrium assumption, like in the linear model, is replaced by a market mechanism that governs the price. The proposed nonlinear model of the market is described by the equations:

$$Q_t^d = \alpha - \beta E_{t-1}[p_t],\ \ \alpha, \beta > 0, \tag{5}$$

$$Q_t^s = \min\{-\gamma + \delta p_{t-1}, r\},\ \ \gamma, \delta, r > 0, \tag{6}$$

$$\Delta p_t = p_t - p_{t-1} = j(Q_t^d - Q_t^s),\ \ j > 0. \tag{7}$$

Substituting equations (5) and (6) into the equation (7) we get first-order nonlinear difference equation which describes dynamics of the price on the considered market:

$$p_t = p_{t-1} + j\left(\alpha - \beta E_{t-1}[p_t] - \min\{-\gamma + \delta p_{t-1}, r\}\right) \tag{8}$$

which depends on nine positive real parameters:
$\alpha, \beta, \gamma, \delta, j, r, \omega, \mu_1, \mu_2$ ($\mu_2 < 1$).

Above equation is a nonlinear difference equation which cannot be solved analytically. Qualitative methods will be used to investigate properties of this model.

Let $F : R_+ \to R_+$ denote the right hand side of the equation (8):

$$F(p_{t-1}) = p_{t-1} + j\left(\alpha - \beta E_{t-1}[p_t] - \min\{-\gamma + \delta p_{t-1}, r\}\right). \tag{9}$$

It is worth mentioning that the expected price depends on p_{t-1} and the right hand side of (9) is well defined. The map F is given by two maps F_i ($i = 1, 2$) defined respectively, in two regions R_i of the phase space:

$$F_1(p_{t-1}) = j(\alpha + \gamma) + (1 - j\delta)p_{t-1} - j\beta E_{t-1}[p_t],$$
$$R_1 = \left\{p_{t-1} \in R_+ : p_{t-1} \leqslant \frac{r + \gamma}{\delta}\right\},$$
$$F_2(p_{t-1}) = j(\alpha - r) + p_{t-1} - j\beta E_{t-1}[p_t],$$
$$R_2 = \left\{p_{t-1} \in R_+ : p_{t-1} > \frac{r + \gamma}{\delta}\right\}.$$

The equation (8) is nonlinear and at the beginning of the analysis equilibria for this system will be determined. Equilibria, sometimes called critical points, are fixed points of the map F, to find all of them it is necessary to find all fixed points of the maps F_i. Fixed points of the map F_i satisfy the following equation:

$$p_{t-1} = p_t = p = const. \tag{10}$$

For the map F_1 equation (10) is equivalent to the equation:

$$(\bar{p}-p)\left[1+\frac{\beta}{\delta+\beta}\left(\bar{w}(\mu_1+\mu_2)-\mu_2\right)\right]=0,$$
$$\bar{w}=\frac{p^2}{p^2+\omega^2(p-\bar{p})^2},$$

which has only one solution $p=\bar{p}$ because equation

$$\mu_2-1=\bar{w}(\mu_1+\mu_2)+\frac{\delta}{\beta} \tag{11}$$

has no solution. Left hand side of (11) is negative ($\mu_2<1$) and right hand side is positive. For the map F_2 equation (10) is equivalent to the equation:

$$\frac{1}{(p-\bar{p})}\left[\frac{\alpha-r}{\beta}-p\right]+\mu_2=\bar{w}(\mu_1+\mu_2)$$

which has only one solution $p^*>\bar{p}$ (fig. 1).

Proposition 2. *Difference equation (8) has one fixed point* $\bar{p}=\dfrac{\alpha+\gamma}{\beta+\delta}$ *for* $p_{t-1}\leqslant\dfrac{r+\gamma}{\delta}$ *and one fixed point* $p^*>\bar{p}$ *for* $p_{t-1}>\frac{r+\gamma}{\delta}$.

In this section we consider fixed points and conditions for which the local asymptotic stability of fixed points is lost. We begin the stability analysis by deriving the first order derivative of the map F (the Jacobian matrix), which is given by the following formula:

$$\frac{dF[p_{t-1}]}{dp_{t-1}}=\begin{cases}1-j\delta-j\beta\frac{dE_{t-1}[p_t]}{dp_{t-1}} & \text{for} \quad p_{t-1}\leqslant\frac{r+\gamma}{\delta},\\ 1-j\beta\frac{dE_{t-1}[p_t]}{dp_{t-1}} & \text{for} \quad p_{t-1}>\frac{r+\gamma}{\delta},\end{cases} \tag{12}$$

where

$$\frac{dE_{t-1}[p_t]}{dp_{t-1}}=\frac{(\mu_1+\mu_2)}{1+\left(\omega\frac{p_{t-1}-\bar{p}}{\bar{p}}\right)^2}\left[1-\frac{2\omega\left(\frac{p_{t-1}-\bar{p}}{\bar{p}}\right)^2}{1+\left(\omega\frac{p_{t-1}-\bar{p}}{\bar{p}}\right)^2}\right]+1-\mu_2.$$

If the absolute value of the derivative (12) evaluated at the equilibrium is strictly less than one then the equilibrium of one-dimensional dynamical system is locally asymptotically stable. It was shown that the fixed point of the linear model is also an equilibrium for the modified model. At this equilibrium the trend followers are predicting perfectly ($w_t=1$) and the derivative (12) calculated at that equilibrium $\bar{p}$, simplifies to:

$$\frac{dF}{dp_{t-1}}[\bar{p}]=1-j\delta-j\beta(1+\mu_1).$$

Proposition 3. *Fixed point* $\bar{p} = \dfrac{\alpha+\gamma}{\beta+\delta}$ *of the difference equation (8) is locally asymptotically stable iff*

$$0 < \delta < \frac{2}{j} - \beta(1+\mu_1).$$

Second fixed point p^* of difference equation (8) will be analysed numerically because there is no analytical formula for this equilibrium. It was shown only that such equilibrium exists.

4.2. Global dynamics and bifurcations

One fundamental characteristics of a complex dynamical system is the possibility of order and chaos, which can exist either separately or simultaneously. In an ordered dynamical system, for arbitrary initial conditions, after going through a transient period the system approaches a periodic behaviour with a predictable periodicity. Chaotic dynamical system exhibits behaviour that depends sensitively on the initial conditions, and long-term prediction is impossible. One characteristic of chaotic motion is sensitivity to initial conditions. Its measure is the largest Lapunov exponent, which is the expotential rate of divergence of nearby orbits in phase space. Theoretically, the Lapunov exponent is negative for systems with stable fixed points or stable cycles and positive for chaos.

Before discussing loss of stability and bifurcations, we need to recapitulate some elementary notions in bifurcation theory necessary in the remainder of this paper. The term bifurcation describes a quantitative change in the orbit structure of a dynamical system, as one or more of the parameters on which it depends is changed slightly. The bifurcation of a fixed point of the map F occurring when its eigenvalue (its derivative evaluated at the fixed point) passes through minus one, the fixed point loses its stability and a stable period-2 cycle is born, this is called a flip bifurcation.

Numerical simulations of the dynamics of the price are provided on bifurcation diagrams. The one dimensional (single parameter) bifurcation diagram for parameter j is presented in figure 1 and the Lapunov exponent over the same interval is presented in figure 2. The parameter j is a reaction parameter, how strongly the market mechanism responds to the imbalance between demand and supply. Both figures suggest that there are three basic types of long-run dynamics. For small values of the parameter j the price is converging to the unique stable stationary equilibrium. At $j \simeq 0.47$ this equilibrium loses its stability through a flip bifurcation and a period-2 stable cycle appears. Then through a cascade of flip bifurcations the model becomes chaotic. When the bifurcation parameter j is increased beyond 0.91 then reversed flip bifurcation can be observed and the dynamics of the model is again periodic.

Figure 1. j – bifurcation diagram

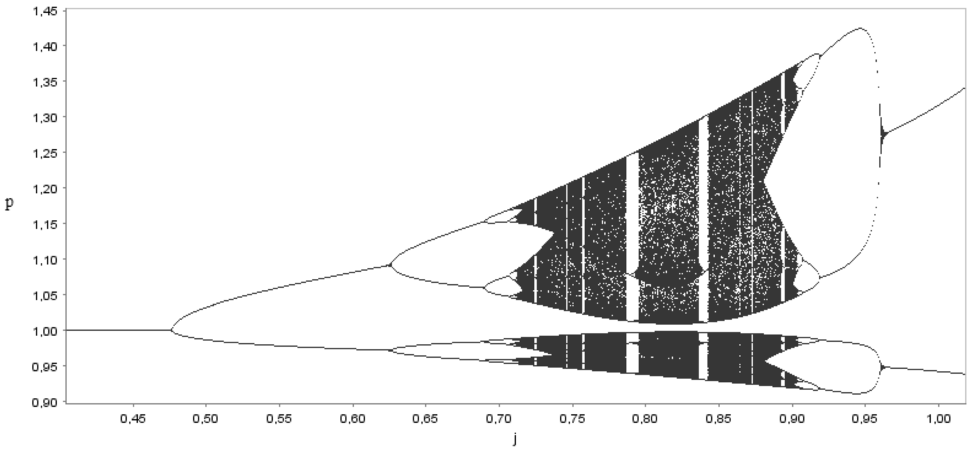


Figure 2. Lapunov exponent

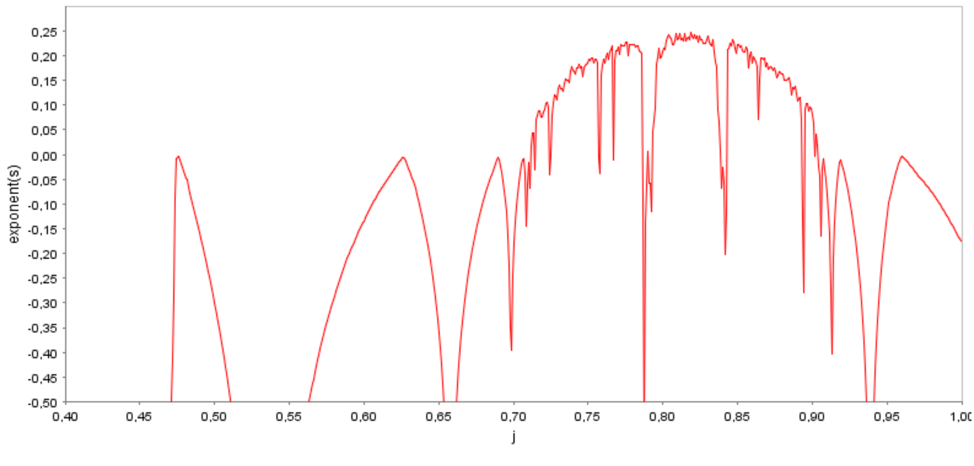


Figure 3 and figure 4 show the long-run dynamics of a model as a multi--function of the parameters α and β respectively which describe the demand side of the market. The α parameter shifts the demand curve. For the low values of this exogenous variable there is a stable period-2 cycle. When $\alpha \in (4.9, 5.3)$ then the model is chaotic. If the bifurcation parameter is increased above 5.3 then reversed flip (or period halving) bifurcation is observed. Price behaviour is periodic. Periodicity is a decreasing function of the α parameter and finally the price is converging to the stable equilibrium where demand equals supply.

Figure 3. α – bifurcation diagram

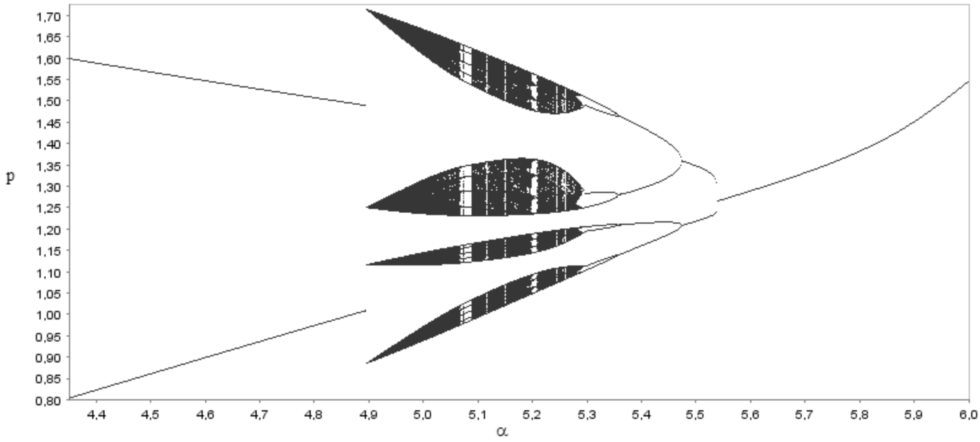
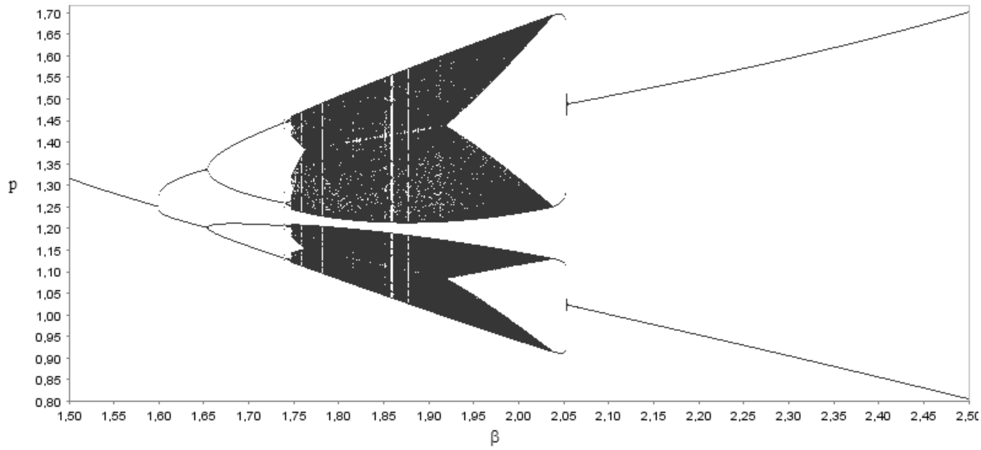


Figure 4. β – bifurcation diagram



Exogenous variable β is a reaction parameter i.e, how strongly the demand responds to the expected price. Figure 4 shows the long run dynamics of the price. For low values of β the market is in equilibrium, demand equals supply. For $\beta \simeq 1.6$ stationary equilibrium becomes unstable as a result of flip bifurcation and a stable period-2 cycle is born. When bifurcation parameter β is increased, at $\beta \simeq 1.75$ the model becomes chaotic and leaves the chaotic zone at $\beta \simeq 2.03$. For $\beta \gtrsim 2.03$ the long-term dynamics are periodic. It is worth mentioning that the amplitude of the periodic attractors is an increasing function of the reaction parameter β.

Figure 5 and figure 6 show the long-run dynamics of the model as a multi--function of the parameters γ and δ respectively which describe the supply side of the market. The γ parameter shifts the supply curve. For low values of this exogenous variable the fixed point of the model is asymptotically stable (figure 5). When $\gamma \in (1.37, 2, 18)$ then the model is chaotic. The chaotic zone is divided by windows of periodic dynamics. If the bifurcation parameter is increased above 2.18 the model leaves the chaotic zone and long-run behaviour of the price is periodic, there is a stable period-2 cycle.

Figure 5. γ – bifurcation diagram

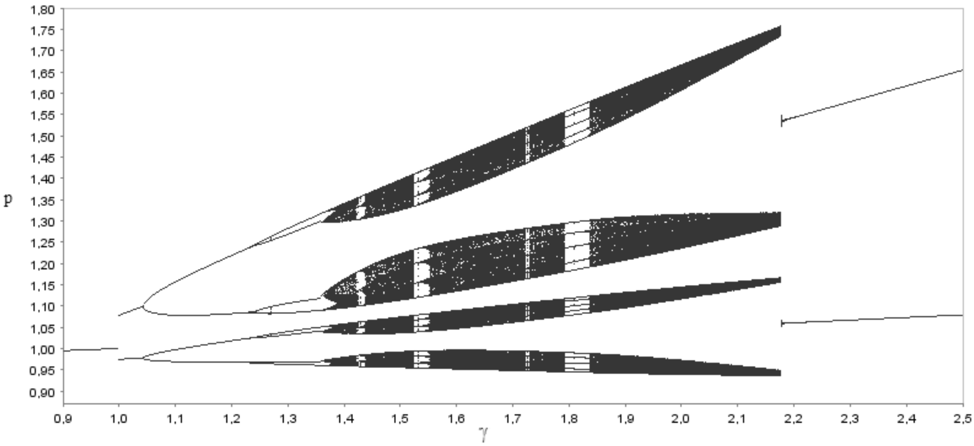


Figure 6. δ – bifurcation diagram

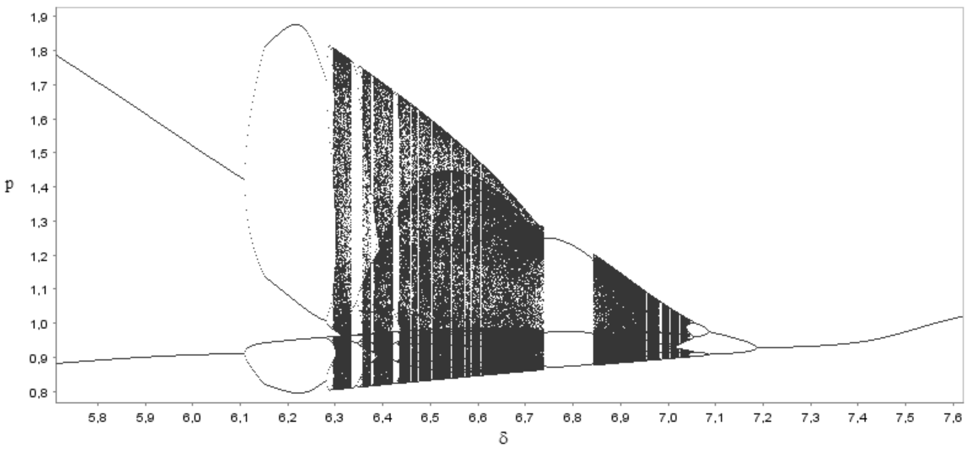


Figure 6 shows the dynamics of the model in relation to the δ parameter which is the slope of supply curve, and suggest the following bifurcation scenario.

If δ is small then there exists a stable period-2 cycle. If δ is increased, then this cycle becomes unstable and flip bifurcations occurs. After infinitely many flip bifurcations the price behaviour becomes chaotic, as δ is increased. A stable period-3 orbit occurs for a small interval of δ, and again the price dynamics become chaotic. When δ is further increased after a cascade of period halving bifurcations there exists stable stationary equilibrium.

4.3. Conclusions

Let us recapitulate the main results of our investigations so far. The proposed nonlinear demand-supply model with expectations has one fixed point (equilibrium). The equilibrium of the linear model is also an equilibrium for the nonlinear model. With the nonlinear model, local stability of a fixed point may be lost while global stability continues in the form of convergence to periodic or chaotic attractors. Introducing expectations into the linear demand-supply model enormously increases the potential complexity of its dynamics. Periodic and chaotic behaviours occur in many possible combinations. The effect of variations of the parameters on stability as well as on the degree of complexity of the dynamics of the system need not be monotonic. Moreover, the proposed model provides sustained and intricate fluctuations of the price which can be observed in economic data.

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Dynamika modelu konkurencji doskonałej z oczekiwaniami po stronie popytowej

Streszczenie

Głównym celem opracowania jest zbadanie wpływu prostego mechanizmu oczekiwań na dynamikę modelu konkurencji doskonałej. Badany model jest nieliniowym układem dynamicznym z czasem dyskretnym. Udowodnione zostanie twierdzenie o istnieniu równowag stacjonarnych. Podane zostaną warunki gwarantujące lokalną asymptotyczna stabilność tychże równowag. W badanym modelu zachodzi bifurkacja podwajania okresu, w wyniku której pojawiają się atraktory okresowe. Opisany zostanie wpływ zmiennych egzogenicznych na dynamikę modelu.

Słowa kluczowe: konkurencja doskonała, oczekiwania, równowaga, bifurkacja, chaos deterministyczny.

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