Roczniki Kolegium Analiz Ekonomicznych
Zeszyt 32/2013

Michał Ramsza

The Bertrand Duopoly with Bounded-Rational Consumers and Explicitly Modeled Demand. A Numerical Example

Abstract

The model presented herein is a variant of the original Bertrand duopoly model. The only difference comes from the more detailed model of behavior of the population of consumer. It is assumed that there is a particular distribution of reservation prices within the population. It is also assumed that consumers use a bounded-rational procedure of choice between the products. The main results are twofold. Firstly, it is shown that it is possible that equilibrium prices are higher than the marginal costs. Secondly, it is shown that there maybe two equilibrium points.

Keywords: Bertrand duopoly, population games.

1. Introduction

Two best known models of duopoly are the Cournot model, cf. Cournot (1838), and the Bertrand model, cf. Bertrand (1883). The Bertrand model concerns the situation with two identical firms setting prices of two identical products simultaneously. Given exogenous demand function satisfying certain mild conditions and some assumptions on the technology employed by the firms the surprising result of the model is that the only Nash equilibrium of the game is a profile of prices $p = (p_1, p_2)$ where $p_i = c$, $i = 1, 2$ and $c > 0$ is a marginal cost (identical for both firms). Thus, firms set prices equal to marginal cost exactly as under the conditions of perfect competition.

The Bertrand model has been extended in many directions over the years. The first problem with the model is the assumption of the constant returns to scale of the production technology employed by the firms. Once this assumption is not satisfied, e.g. a firm cannot meet the whole demand, the result does not hold. The other problem, with a huge body of literature, is the lack of collusion between the firms. Once there is some kind of collusion the original result is again not true. An equally large body of literature has been devoted to the differentiation of products and consequences of this process.

This paper is concerned with a more in-depth model of the very demand. In the standard Bertrand model the demand function is given exogenously. Additionally, it is assumed that even the slightest difference in prices of the goods leads to the situation where the cheaper firm gets the whole market. In this paper the starting point for the demand function is the distribution of reservation prices. Also, it is assumed that consumers are only bounded-rational in the sense that if given two goods with prices $p_1 < p_2$ they buy the cheaper product more often (with higher probability). These kind of assumptions have been used before in (Ramsza, 2010, chpt. 7), and in (Olesiński, 2012). However in (Ramsza, 2010) the analysis was concerned with goods of interdependent characteristics. The model proposed in (Olesiński, 2012) is closer to the model presented herein but the models differ in the tools used.

The rest of the paper is organized as follows. Section 2 introduces the model of the demand. Section 3 is concerned with firms' payoffs. The main analysis is presented in section 4. Section concludes.

2. Demand – a model of a population of consumers

Any demand function must stem from the fundamental question: *how many consumers can afford (or want) to buy a product?* This is usually modeled through reservation prices r_j, where j is an index of a consumer. If $r_j \geqslant p_i$ then the consumer j can afford to buy the i -th product. If $r_j < p_i$ then the consumer j does not buy the product i. The demand function is a results of a consumers population having various reservation prices. It may be assumed that the reservation prices r_j are realizations of independent, identically distributed random variables R_j with a cumulative distribution function denoted by R with some abuse of notation. It is assumed that the support of the distribution R is a set of nonnegative real numbers.

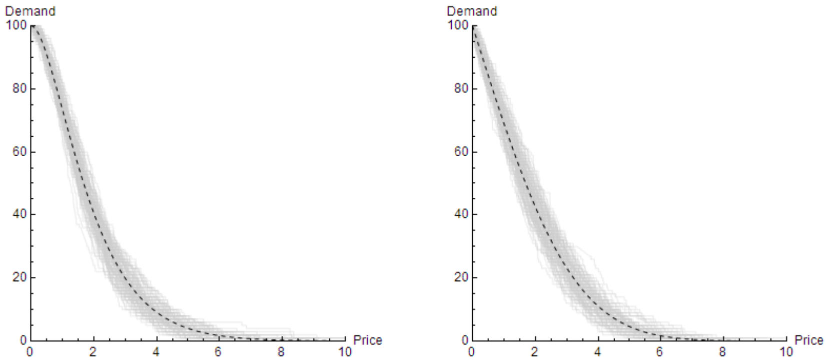
Figure 1 shows examples of two demand functions that are results of a population of 100 consumers having their reservation prices given according to different distributions. Figure 1(a) shows the demand function if the reservation prices are drawn from the gamma distributions with parameters $(1, 2)$ and figure 1(b) shows the distribution function if the reservation prices are drawn from the half-normal distribution with the parameter $1/2$. Each realization of the demand function has typical properties usually assumed in theoretical economics.

The demand function tells how many consumers buy a product at a given price. If there are two similar products at prices p_1 and p_2 respectively, additional mechanism of consumer's choice is required to know how many consumers buy a product of a given firm at a given profile of prices. In the original Bertrand model it is assumed that consumers react to even the slightest difference in prices. This is highly questionable. Instead it is assumed herein that consumers have only limited capability or will, to react to small differences in prices. If the first product is cheaper a consumer buys this product with a higher probability than the other product. The larger the price difference the larger the disproportion in probabilities. This behavior is modeled through the smoothed best reply, cf. D. McFadden (1981). In particular the conditional probability that a consumer buys the first product given the profile of prices $p = (p_1, p_2)$ is given as $\Psi_\sigma(p_2 - p_1)$, where Ψ_σ is the cumulative distribution function of the normal distribution $\mathrm{N}(0,1)$. The parameter σ measures the rationality level of a consumer. The lower the value of $\sigma > 0$ the more rational the consumer. In the limit $\sigma \to 0$ the standard best reply function is recovered.

Figure 1. Demand functions from various distributions. The dashed line is the average demand function. The gray lines show realization of the demand functions. Both functions for the population of 100 consumers.

(a) The demand function from the gamma distribution $\boldsymbol{\Gamma(1,2)}$

(b) The demand from the half-normal distribution with the parameter 1/2



3. Payoffs – profits of firms

As in the original model there are assumed two firms with constant and identical marginal costs c_i which for simplicity are set to zero, $c_i = 0$, $i = 1, 2$. Firms simultaneously set prices p_i, $i = 1, 2$. Payoff of the i-th firm is given as

$$\pi_i(p) = D_i(p)(p_i - c_i) = D_i(p)p_i,$$

where $p = (p_1, p_2)$ is the profile of prices. The demand $D_i(p)$ is a random variable though thus firms maximize the expected profit $\max\limits_{p_i \geqslant 0} \mathrm{E}D_i(p)p_i = \max\limits_{p_i \geqslant 0} p_i\mathrm{E}D_i(p)$.

The average payoff depends on the profile of prices $p = (p_1, p_2)$. Let b_i be the probability of the event that a consumer buys from the i -th firm, $i = 1, 2$ and let b_0 be the probability of the event that a consumer cannot afford to buy any goods.

For the case $p_1 < p_2$ it is

$$\begin{aligned} b_0 &= \Pr\left[R < p_1\right], \\ b_1 &= \Pr\left[p_1 \leqslant R < p_2\right] + \Pr\left[p_2 \leqslant R\right] \Psi_\sigma(p_2 - p_1), \\ b_2 &= \Pr\left[p_2 \leqslant R\right] (1 - \Psi_\sigma(p_2 - p_1)). \end{aligned}$$

Symmetrically, for the case $p_2 < p_1$ it is

$$\begin{aligned} b_0 &= \Pr\left[R < p_2\right], \\ b_1 &= \Pr\left[p_1 \leqslant R\right] \Psi_\sigma(p_2 - p_1), \\ b_2 &= \Pr\left[p_2 \leqslant R < p_1\right] + \Pr\left[p_1 \leqslant R\right] (1 - \Psi_\sigma(p_2 - p_1)). \end{aligned}$$

The reservation prices for consumers are assumed to be independent with the identical CDF R thus the expected profits for the firms are given as

$$\mathrm{E}\left[\pi_1(p)\right] = \begin{cases} n\left(\Pr\left[p_1 \leqslant R < p_2\right] + \Pr\left[p_2 \leqslant R\right] \Psi_\sigma(p_2 - p_1)\right) p_1 & \text{for } p_1 < p_2, \\ n\left(\Pr\left[p_1 \leqslant R\right] \Psi_\sigma(p_2 - p_1)\right) p_1 & \text{for } p_1 \geqslant p_2, \end{cases}$$

for the first firm and

$$\mathrm{E}\left[\pi_2(p)\right] = \begin{cases} n\left(\Pr\left[p_2 \leqslant R\right] (1 - \Psi_\sigma(p_2 - p_1))\right) p_2 & \text{for } p_1 < p_2, \\ n\left(\Pr\left[p_2 \leqslant R < p_1\right] + \Pr\left[p_1 \leqslant R\right] (1 - \Psi_\sigma(p_2 - p_1))\right) p_2 & \text{for } p_1 \geqslant p_2, \end{cases}$$

for the second firm. It is clear the the value n does not play any role in the further analysis and is set to $n = 1$. The formulas for the expected payoffs of the firms complete the description of the game.

Figure 2. The profit function of the first firm

(a) Graph of the profit function of the first firm **(b) Contour map for the same profit function**

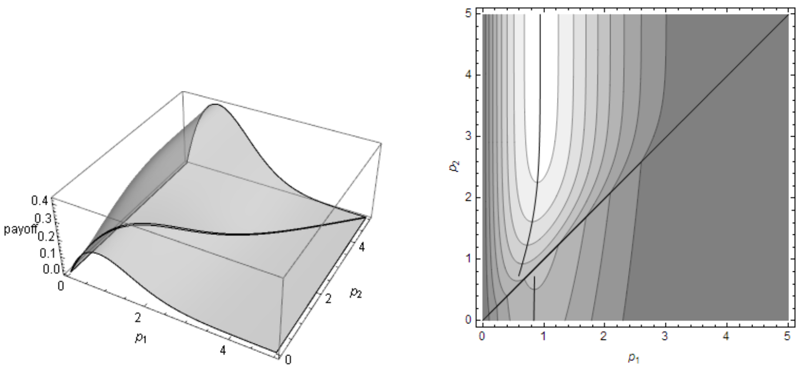


Figure 2 shows an example of the profit function of the first firm for the demand generated from the half-normal distribution with parameter $\theta = 1$, $\sigma = 5$ and contour map for the same profit function. On both figures a line $p_1 = p_2$ is marked.

4. Analysis and discussion

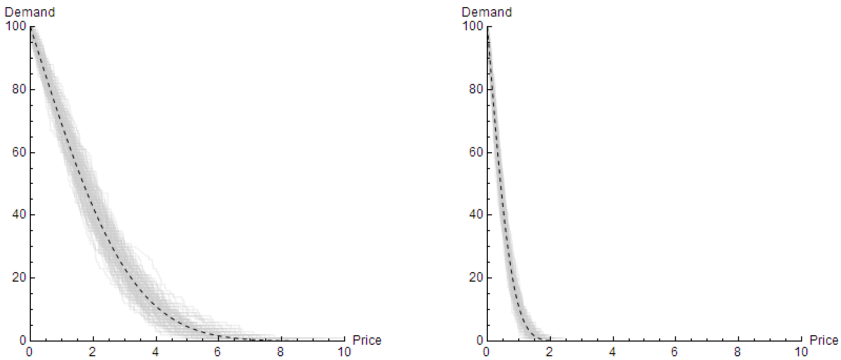
The game defined in sections 2 and 3 is too general to be solved analytically in general. Also, this is not the aim of this paper to attempt to solve the model analytically. Instead, it will be shown by a numerical example that it is possible to get two results.

Firstly, it will be demonstrated that it is possible to have an equilibrium at which prices set by the firms are higher than marginal costs thus making it different from the original result. Secondly, it will be demonstrated that in fact there are two equilibrium points for a wide range of parameters' values.

For the numerical example the CDF R is taken to be the half-normal distribution with parameter θ . The interpretation of the parameter σ is as before. The interpretation of the parameter $\theta > 0$ is that for small values of this parameter the demand is high even for high prices while for large values the demand falls very quickly forcing any equilibrium prices to be close to the marginal costs c_i. Figure 3 shows two examples of the demand functions derived from the half-normal distribution for $\theta = 1/2$ and $\theta = 2$.

Figure 3. Demand functions derived from the half-normal distribution with parameters $\boldsymbol{\theta}$

(a) $\boldsymbol{\theta = 1/2}$ (b) $\boldsymbol{\theta = 2}$



The expected payoffs of the firms for the case of the half-normal distribution read

$$\mathrm{E}\left[\pi_1(p)\right] = \begin{cases} \frac{1}{2}p_1\left(2\mathrm{erfc}\left(\frac{\theta p_1}{\sqrt{\pi}}\right) + \mathrm{erfc}\left(\frac{\theta p_2}{\sqrt{\pi}}\right)\left(\mathrm{erfc}\left(\frac{p_1-p_2}{\sqrt{2}\sigma}\right) - 2\right)\right) & \text{for } p_1 < p_2, \\ \frac{1}{2}p_1\mathrm{erfc}\left(\frac{\theta p_1}{\sqrt{\pi}}\right)\mathrm{erfc}\left(\frac{p_1-p_2}{\sqrt{2}\sigma}\right) & \text{for } p_1 \geqslant p_2, \end{cases}$$

for the first firm and

$$\mathrm{E}\left[\pi_2(p)\right] = \begin{cases} \frac{1}{2}p_2\mathrm{erfc}\left(\frac{\theta p_2}{\sqrt{\pi}}\right)\mathrm{erfc}\left(\frac{p_2-p_1}{\sqrt{2}\sigma}\right) & \text{for } p_1 \leqslant p_2, \\ \frac{1}{2}p_2\left(\mathrm{erfc}\left(\frac{\theta p_1}{\sqrt{\pi}}\right)\left(\mathrm{erfc}\left(\frac{p_2-p_1}{\sqrt{2}\sigma}\right) - 2\right) + 2\mathrm{erfc}\left(\frac{\theta p_2}{\sqrt{\pi}}\right)\right) & \text{for } p_1 > p_2, \end{cases}$$

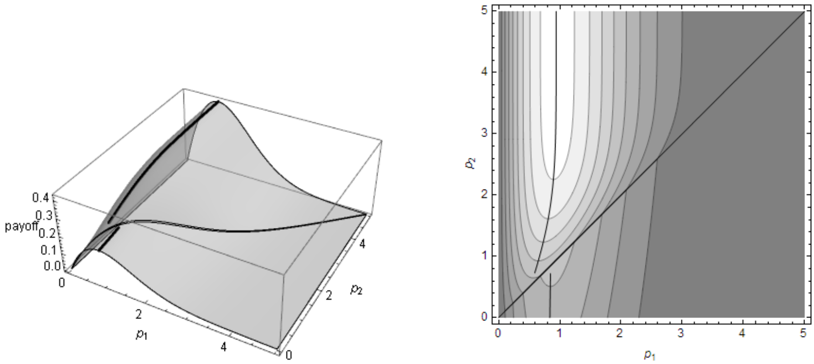
for the second firm, where

$$\operatorname{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2} dt$$

is the complementary error function.

Figure 4. Profit function and contour map with the best reply mapping overlaid

(a) Profit function **(b) Contour map**



The best reply mapping (reaction curve) is derived through numerical optimization of the optimization problem of the form

$$\max_{p_i \geqslant 0} \mathrm{E}\left[\pi_i(p_i, p_{-i})\right]$$

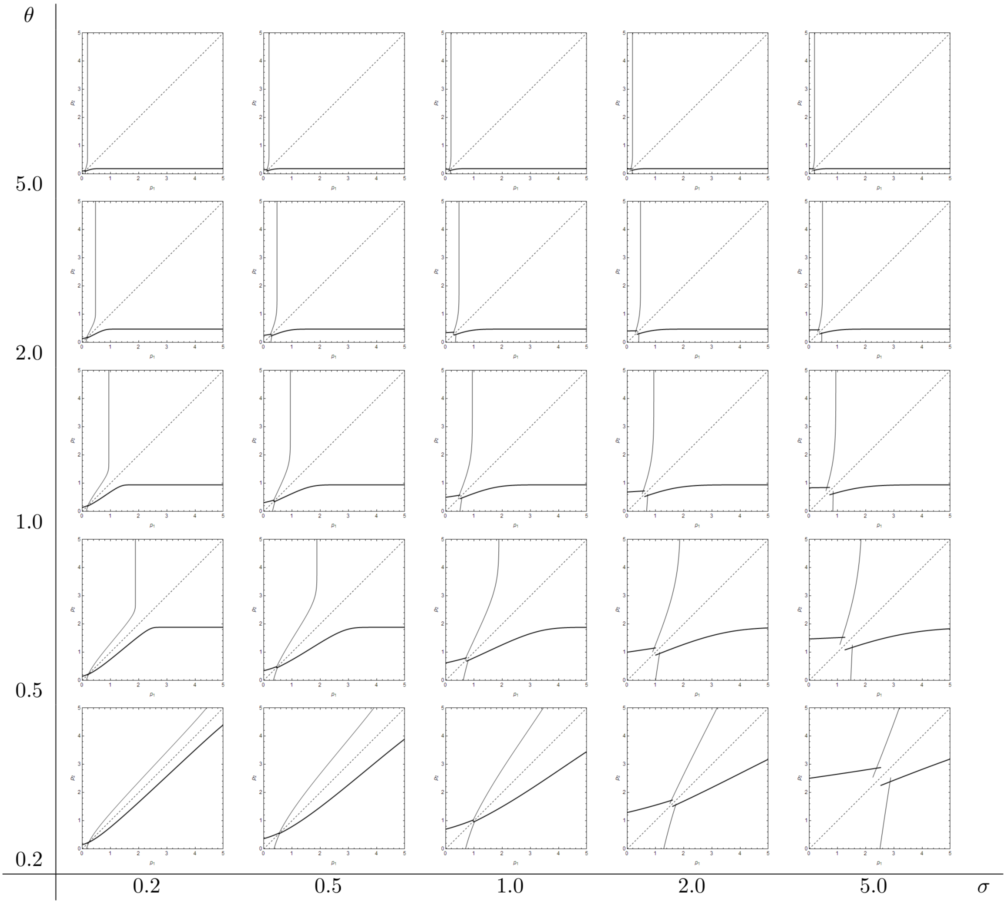
for each firm $i = 1, 2$, where $-i$ denotes the other firm. The particular optimization algorithm employed was the Nelder-Mead algorithm (the so called "simplex" method), cf. Nelder and Mead (1965). Figure 4 shows the profit function of the first firm and the contour map of the profit function with the best reply mapping overlaid.

Figure 5 shows reaction curves for various combinations of parameters θ and σ. Two things, mentioned before, can be observed. Firstly, the equilibrium prices are always strictly positive, thus, above the marginal costs. For θ large the demand function falls quite fast and forces the equilibrium prices to be close to zero, but on closer inspection the actual equilibrium prices are positive.

Secondly, there are two equilibrium points for a wide range of the parameter's values. In particular for larger values of σ and smaller values of θ it is especially obvious.

The observed departures from the results of the original model are due mainly to the introduction of the bounded-rational behavior of players. Once one of the firms sets the price equal to the marginal cost the other firm can slightly increase the price and still face the positive demand due to the random choice of the consumers. The level of this demand depends crucially on the value of the parameter σ. For small values of this parameter the equilibrium prices are close to the marginal costs.

Figure 5. Reaction curves for various combinations of values of parameters θ (rows) and σ (columns)



The particular shape of the distribution of the reservation prices has some influence on the particular position of equilibrium points. However, the strongest influence is through the general rate decrease of the derived demand function. Quickly falling demand further mitigates the difference between the original model and the presented model. On the other hand, if the demand function allows for a larger demand at higher prices it leads to larger discrepancies between the original model and the model presented herein.

5. Conclusions

The paper concerns the variant in the original Bertrand model of price duopoly. In the model considered the demand function was derived directly from the consumer population with a particular distribution of reservation prices. Also, it was assumed that consumers exhibit some degree of bounded-rationality as far as the choice between the goods is concerned.

With this assumptions it was shown that two new situations occur. Firstly, the equilibrium prices may by strictly larger than the marginal costs, thus, to a degree resolving the Bertrand paradox. Secondly, it is possible to get two equilibrium points. This maybe of some importance for dynamic models, especially the ones in discrete time because it may lead to oscillations.

The model proposed herein is just a numerical example. Further research may involve analytical study of the presented model and a dynamic model of two competing firms with a population of consumers.

Bibliography

[1] Bertrand J. (1883), *Théorie mathématique de la richesse sociale*, "Journal des Savants" 67, pp. 499-508.

[2] Cournot A. (1838), *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, (*Researches into the Mathematical Principles of the Theory of Wealth*), N. Bacon, London: Macmillan, 1897 (wersja angielska).

[3] McFadden D. (1981), *Econometric models of probabilistic choice*, in C.F. Manski & D. McFadden, *Structural analysis of discrete data with econometric applications*, Cambridge: MIT Press, pp. 198-272.

[4] Nelder J.A. & Mead R. (1965), *A simplex method for function minimization*, "The Computer Journal" 7, pp. 308-313.

[5] Olesiński B. (2012), *The dynamics of Bertrand model with multi-agent demand and bounded-rational firms*, B.Sc. Thesis, Warsaw School of Economics.

[6] Ramsza M. (2010), *Elementy modelowania ekonomicznego opartego na teorii uczenia się w grach populacyjnych*, Oficyna Wydawnicza SGH.

***

Duopol Bertranda z jawnym modelem popytu. Analiza numeryczna

Abstrakt

W pracy jest zaprezentowany wariant modelu duopolu Bertranda. Zaprezentowany model różni się od modelu standardowego tym, że w sposób jawny został wymodelowany popyt na produkowane dobra w populacji konsumentów w oparciu o założony rozkład cen rezerwacji. Dodatkowo konsumenci posługują się procedurą wyboru, która nie jest w pełni *racjonalna a jedynie przybliża w wyidealizowaną najlepszą odpowiedź. Dwa głowne wyniki pracy to wskazanie, że nawet przy standardowych założeniach w równowadze ceny są wyższe od kosztów krańcowych oraz możliwość występowania wielu cen równowagowych.*

Słowa kluczowe: Duopol Bertranda, gry populacyjne.

Author:

Michał Ramsza, Department of Mathematics and Mathematical Economics, Warsaw School of Economics, Al. Niepodległości 162, 02-554 Warsaw, Poland,
e-mail: michal.ramsza@gmail.com