Mikołaj Jasiński

The terms of cooperation's stability.
What is the reason of flow between coalitions?

Abstract

The following article presents an application of game-theoretical models, meant to describe and explain the conditions of stability of coalitions created within decision-making bodies. The basis of the presented approach is the concept of power indices, as well as subadditivity and superadditivity of coalitions. The article also illustrates how to apply the model of oceanic games for analyzing assemblies consisting of many participants. These studies are exemplified by the results of research on the decision-making processes in the Sejm (the lower house of the Polish parliament) and among delegates of the 2008 U.S. presidential nominations of the Democratic Party, which lead to clear, intuitive interpretations, useful for explanations of processes occurring within the decision-making bodies.

Keywords: coalition, Shapley-Shubik power index, weighted majority game, oceanic game, sub- and superadditivity of a coalition.

1. Introduction

Natural elements of the decision-making process within various assemblies, both political (parliaments or assemblies of electors) and economic (shareholders meetings or company management boards) are agreements, expanding the number of participants and finally breaking the agreement. Therefore, the analysis of the terms of coalition's stability is a matter of primary interest for researchers dealing with phenomena occurring within decision-making bodies.

The theory of cooperative games offers particularly convenient implements for analyzing these phenomena, both useful and easily interpretable. An important aspect of the game-theoretical analyses is the number of players. One must differentiate between two cases; in the first one a collective decision is made by a small group of people, e.g. several members of a government, corporation management, or the Council of the European Union, where each decision-maker can distinguish between individual voters. The second case occurs when a decision-maker remains in the crowd of anonymous individuals, which in fact happens quite frequently during party primary elections, referendums, general elections, or even in highly fragmented parliaments of management boards.

There are numerous approaches to the decision-making issues within large assemblies (e.g. Auman & Shapley, 1974; Ekes, 2003; Malawski, 2000; Milnor

& Shapley, 1961; Wieczorek, 2004, 2005). Milnor and Shapley presented an oceanic games concept, which deserves special attention (1961). In a series of short texts (e.g. Shapiro& Shapley, 1960; Milnor & Shapley, 1961; Shapley, 1961) complemented by a comprehensive monograph (Auman & Shapley, 1974), Shapley and his co-operators present the principles and properties of the game models with a large group of players presented as infinitesimal intervals (measure-zero sets) on a unit interval[1]. In this article I shall study examples of applying the classical approach of the cooperative games theory and oceanic games to the analysis of the decision-making bodies. Shapley-Shubik index is only one index of power that satisfies the block postulate[2] (e.g. Falsenthal & Machover, 1995; Jasiński, 2000). This postulate is one of the crucial properties necessary for power indices to be useful for analysis of blocks' expansion mechanisms. I shall focus on the attained results and essential conclusions, while limiting the presentation of the formalization to the necessary minimum, especially since slightly more detailed formalized assumptions of the oceanic games have already been discussed in a separate article (Jasiński, 2009).

2. Oceanic games. A short presentation of the model

The scheme below shows how during our further discussion I shall write down the oceanic weighted majority game with decisive quote-rule c, m-person (so-called atomic[3]) players of "weights": $w_1, w_2, ..., w_m$ and ocean of total weight α (continuum of players of measure-zero weights) representing a very large number of "minor" players:

$$[c; w_1, w_2, ..., w_m; \alpha].$$

I shall also refer to the solutions of the oceanic weighted majority games characterized by the majority rule $c = 1/2$, with one major player of weight x, as well as to games with two major players having weights x and y, and in each case one ocean of total weight $\alpha = 1-x$ (games with one major player) or $\alpha = 1-x-y$ (games with two major players), which means games of type, respectively:

$$[1/2; x; \alpha] \text{ and } [1/2; x, y; \alpha].$$

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[1] In fact, a certain measurable space is considered (I, C) isomorphic with ([0,1], B) – measurable space generated by the Borel subsets of the closed unit interval. It is of no significance for the correctness of the conclusions presented. Since the text is of rather application character, I still try to avoid excessive formalization.

[2] If one of the players joining a block is a significant player, the power index of the bloc should be greater than the power index of the other player alone.

[3] The notion of an "atomic player" refers directly to the concept of indivisibility (Greek: *atomos*). "Atom" shall be understood in game v as such a non-zero coalition S (and without losing the generality treated as not empty), that $\forall T \subset S$ or T – zero coalition.

Shapley together with Aumann (1974), as well as with Milnor and Shapiro (1960, 1961) has proved several useful theorems enabling us to determine solutions of oceanic games with any number of major players and many oceans. These works triggered a series of suggestions concerning the oceanic games model used for social purposes – in political science and economy (i.a. Straffin, 1977 and 1982; Jasiński, 2009).

In the games presented below, Shapley-Shubik index values of players x and y are described by the following expressions, depending on the size of the total weight of the major players, respectively, x or $x + y$:

For game $[1/2; x; \alpha]$:

$$\text{for } x \leqslant 1/2 : \varphi_x = \frac{x}{1-x},$$
$$\text{for } x \geqslant 1/2 : \varphi_x = 1.$$

Total power of the ocean equals $\Phi = 1 - \varphi_x$.

For game $[1/2; x, y; \alpha]$:

$$\text{for } x + y \leqslant 1/2 : \varphi_x = \frac{x(1-x-2y)}{(1-x-y)^2}, \varphi_y = \frac{y(1-x-2x)}{(1-y-x)^2},$$
$$\text{for } x + y \geqslant 1/2 : \varphi_x = \frac{\left(\frac{1}{2}-y\right)^2}{(1-x-y)^2}, \varphi_y = \frac{(1-x)^2}{(1-y-x)^2}.$$

Total power of the ocean equals $\Phi = 1 - \varphi_x - \varphi_y$.

3. 2008 primary elections in the USA. Applying oceanic games to political analysis – bandwagon effect model. New questions

One of the most interesting applications of the oceanic games model is explaining the mechanisms of a phenomenon known in the area of political science as "bandwagon effect"[4], which means shift towards the stronger.

In order to explain this, let us imagine an electoral competition between a few or just two opposing candidates. Each of them has a separate group of committed voters. However, there is also a group of uncommitted voters who have not yet decided whom to support or are ready to change their commitment. Often this group is large enough to determine who will be the winner. In this case the bandwagon effect could happen.

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[4] The name of the phenomenon discussed here comes from and idiom "climb on the bandwagon". This expression refers to an old American practice of voters demonstrating their support of a politician by climbing the decorated platform of his cart, where a band of musicians was playing, promoting the politician.

Uncommitted voters often do not care who the winner will be. Still, they are interested in joining the winner-to-be as soon as possible, if only for the reason of expecting a future "reward" from the winner for supporting him. But, is just any advantage of the more powerful electorate enough to trigger the in-flow of uncommitted voters?

The research shows that it is not true – if it was, the result of any political competition would then always be pre-determined – at the moment it becomes clear who has more supporters. So, is there any threshold to when this escape starts? In what conditions do the indecisive voters remain uncommitted, even if they would rather not "waste" their vote?

Example No. 1. Obama and Clinton

This problem has been discussed in great detail in my article published in "Decisions" (Jasiński, 2009) and illustrated by the data from the 2008 Democratic Party primary election in the USA preceding the presidential election.

Nomination of Barack Obama was not obvious in 2008 – it was even quite improbable in the beginning. A model constructed on the basis of the oceanic games made it possible to predict and explain the mechanism of the uncommitted delegates (superdelegates and unpledged delegates) joining the supporters of the Illinois senator.

Two terms that I have used need to be explained here: *superdelegates* and *pledged delegates.* Primary elections are of indirect character. Traveling through the country, during meetings (primaries and caucuses) the politicians solicit for the support of the party delegates assigned to individual states, as well as of the members of the party establishment. The result of the primary election in a given state influences the number of delegates from this state supporting a politician. In fact, a fraction of the state delegates are obliged to support the candidate who won the primary election in their state. These are the so-called *pledged delegates.* The Democrats accepted a principle according to which the candidates receive votes of these delegates proportionally to the support won during primary election. While members of the party establishment of a given state (congressmen, governors, former presidents etc.) are called *superdelegates* and they can vote freely, according to their own fancies. They are also the most prone to "upward business cycle" and to calculations. The total support of all delegates leads to a decision made during a party convention about who shall be the party candidate for the election.

In the same article I have also pointed out that this model appeared to be not sufficient to explain another, not less interesting phenomenon of confirmed Hilary Clinton supporters (*pledged delegates*) escaping to Obama's camp, because it meant not only "drawing in" the uncommitted voters, but also decomposition

of a bloc that so far used to be stable. Interesting attempts were made in order to apply the oceanic games to constructing a model that would explain this occurrence as well (Straffin, 1982). Here I shall present the possibilities and limitations of accessible solutions, as well as suggestions for complementing this model.

First of all however, it is essential to present briefly the game-theoretical model, which explains the conditions of the bandwagon effect. (This shall enable me to describe the model showing the mechanism of Clinton's electorate decomposition in the memorable primary election). Referring to the concept of oceanic games allowed for developing circumstances to determine the flow of individual voters (minor players from the ocean) to one of the major players competing for influences within an assembly.

Figure 1. Flow regions and region of stability of the ocean with points marking the support for Barack Obama and Hillary Clinton from February 2008 to June 2008
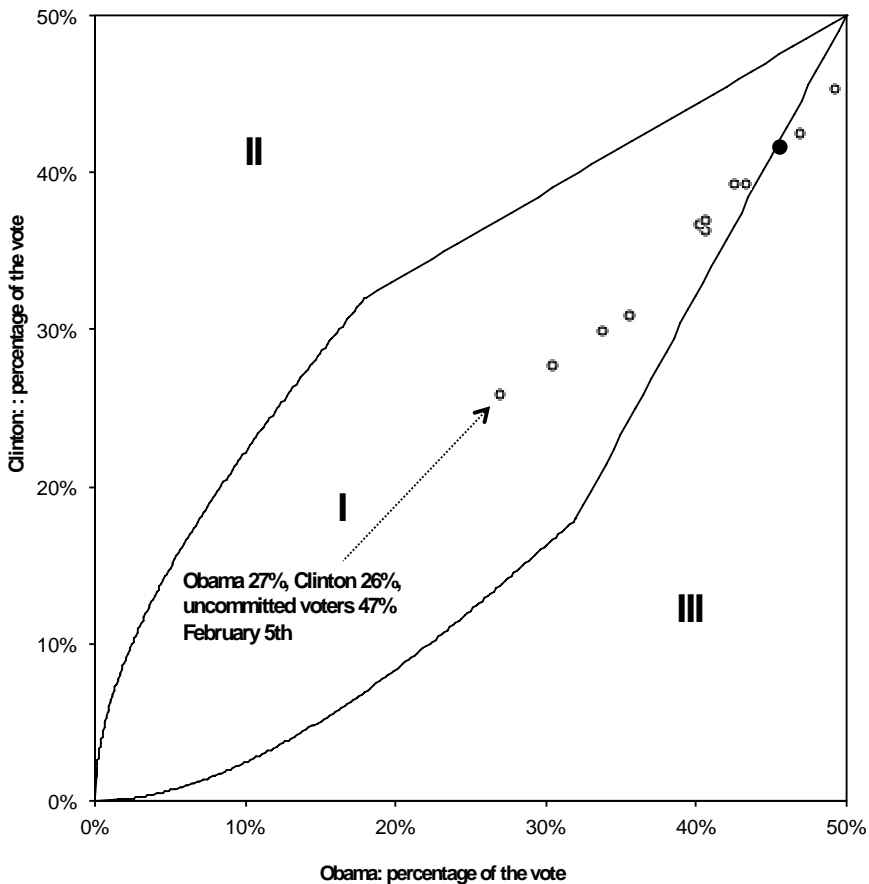
We are considering an oceanic game with two major players – type $[1/2; x, y; \alpha]^5$. The figure 1 presents the conditions determining their weights, which we expect the rational minor players to use to decide if it is worth joining one of the bigger players and the "flow" will start. These conditions are represented by the three regions below.

Both coordinates in Figure 1 present in percents the weight (proportion) of each major player's vote (horizontally: percentage of the vote for Obama, vertically – for Clinton). The points of the coordinate system represent thus different situations of players x and y (Obama and Clinton), depending on the players' weights. For example the bottom point on the figure indicates the fractions of votes supporting both competitors on February 5th. The shape that appears in the figure is limited by the curve determined on the basis of applying the model called the bandwagon curve.

- Region I (limited by this shape) represents these weights of the major players, for which the condition of the ocean flow "to the bigger" is not fulfilled.

- Region II shows configurations of the votes cast for each of the two players in which we anticipate the flow to the player marked on the vertical coordinate (here, this is Clinton's vote).

- Region III presents situations in which we anticipate the ocean flow to the player whose support is marked on the horizontal coordinate (here, Obama).

Also, the figure includes points illustrating results of the vote obtained in the succeeding phases of the primary election by Barack Obama and Hillary Clinton. The points show the percentage of voters supporting each of the politicians during the last 5 months of the primary elections. In February (the first point from the bottom) both Obama and Clinton had an equal chance. During the next few months, Obama's advantage over his rival was slowly but systematically increasing. The situation changed on May 6th – from that day Obama had not only been gaining more votes in the states where he was already winning, but also the superdelegates committed to Clinton so far, started to join Obama's bloc.

The number of delegates supporting Clinton finally decreased from over 1926 to 1896. It should be emphasized that during the electoral campaign it happens extremely rarely, that a delegate (even the superdelegate) who already voted for a certain candidate would shift his or her commitment to the other one.

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^{5} In fact, this situation refers to a modified game, which I call "a quarrel of sole competitors game" (QSC). This type of game excludes coalitions that would be created by both competing rivals – major players. Therefore, for a QSC game the Shapley-Shubik power index expressions slightly differ from those of an oceanic game presented above in its classical form (Jasiński, 2009: p. 46).

The observations mentioned above shall become more understandable when one notices that the point representing the first May listings (May 6th) is the first black marked point located behind the bandwagon curve. Therefore, it appears that this mass flow of the uncommitted delegates to the Illinois senator's bloc recorded at that time, is an example of the described above bandwagon effect – mass escape to the stronger camp, in other words – flow of the rational ocean to the major player. It is also worth mentioning that since May 15th Obama had gained support of stable delegates committed to his rival so far, the pledged delegates. Explaining this occurrence would require complementing the bandwagon effect model with the description of conditions determining the choice of optimal bloc and stability of agreement.

4. Potential blocs and permanent coalitions, or profitability of agreements – a classical approach

Analysis of sub- and superadditivity based on power indices constitutes a natural tool for evaluating the profitability of a coalition. If the total of the indices for the players who could create a bloc together is larger than the index value of this potential bloc, then this is a strong argument against transforming the potential bloc into a permanent coalition (Straffin, 1982, p. 274; Jasiński, 2000, p. 73). A decision about uniform voting is influenced by many factors, like immediate interests, ideological sympathy, voters' expectations (in case of a potential parliamentary bloc or a group of councillors), etc. However, we can say that the interests resulting from the natural need of participating in winning coalitions will have a disintegrating effect on the potential bloc and will not foster permanence of such an agreement.

A completely different situation will take place, however, when after joining the votes of the bloc, its power index will appear larger than the total of indices of the voters forming this bloc. We should use the notion of superadditivity here. Among the case studies concerning the formal theory of decision, especially worth mentioning is the text by Marek Kamiński (1998), in which the author analyses the consolidation of the Polish party system in the years 1993-97. Kamiński, among others, writes about the determinants of the effective uniting process of the right wing, which at that time remained outside the Parliament, into a coalition Solidarity Electoral Action (*Akcja Wyborcza Solidarność* – AWS). Kamiński, however, employed the above-mentioned notions in a slightly different sense, because the parties co-forming AWS did not participate in the parliamentary game. Discussing the sub- and superadditivity, Kaminski referred to the electoral mechanisms, comparing the number of mandates won by a coalition acting together to the number of mandates that each of the coalition partners could win in case if

they decided to act separately. Firstly, I will study the election result for the Sejm of the 3rd term of office (1997–2001), by applying the Shapley-Shubik power index values, which appears to be the right tool for examining the coalition stability, since this index fulfils the bloc postulate (Jasiński, 2000).

Example No. 2. AWS in the Sejm of the 3rd term of office

The history of the AWS can not only serve as a great example of the very first Polish political success of a coalition, but also as a case of the most spectacular collapse of an initially extremely stable political bloc. What could be the explanation of the incredible unity and endurance of the AWS coalition and its final abrupt dissolution into smaller groups? How to explain consolidation process of the political environments that emerged from the AWS and Union of Freedom (*Unia Wolności* – UW), which finally resulted in generating the parties that today dominate the Polish political scene: Civic Platform (*Platforma Obywatelska* – PO) and Law and Justice (*Prawo i Sprawiedliwość* – PiS)? On hundreds of pages the politicians, commentators and journalists have dealt with different and often conflicting explanations of these phenomena. These highly casuistic commentaries deserve to be complemented by the analysis referring to the formal models described above.

After the 1997 election the structure of the Polish Sejm presented itself as follows:

Table 1. Sejm of the Republic of Poland – beginning of the 3rd term

Club or parliamentary circle	No. of mandates
Solidarity Electoral Action (AWS)	201
Democratic Left Alliance (*Sojusz Lewicy Demokratycznej* – SLD)	164
Union of Freedom (UW)	60
Polish Peasant Party (*Polskie Stronnictwo Ludowe* – PSL)	27
Movement for the Reconstruction of Poland (*Ruch Odbudowy Polski* – ROP)	6
German minority MPs (non-attached MPs)	2

MPs of the AWS founded, in fact, one parliamentary club, but they kept their party affiliations – AWS was, in fact, a coalition of several parties and the table 2 presents the party structure of the AWS Parliamentary Club (AWSPC).

It occurred, that despite such a big dissipation, for most of the 3rd term AWS MPs voted in unison, and the club survived for a long time in an unchanged form (it appears that the secession of two MPs fractions – one grouped around Adam Słomka and another one around Jan Łopuszański did not influence the stability

of the AWSPC). During the key voting approximately 95% of AWS MPs voted in unison[6]. This club was characterised by only slightly higher level of discipline in comparison with the other, much smaller member of the coalition – the UW Parliamentary Club (UWPC), which was composed of MPs mainly from a single party (UW).

Table 2. AWSPC – beginning of the 3rd term

Affiliation	No. of mandates
Solidarity Electoral Action (AWS)*	110
The Christian National Union (*Zjednoczenie Chrześcijańsko-Narodowe* – ZChN)	27
The Conservatists (The Conservative People's Party – *Stronnictwo Konserwatywno-Ludowe* or SKL, Conservative Coalition – *Koalicja Konserwatywna*: KK, and others)	18
Centre Agreement (*Porozumienie Centrum* – PC)	14
Confederation of Independent Poland (*Konfederacja Polski Niepodległej* – KPN)**	9
Christian Democrats Party (*Partia Chrześcijańskich Demokratów* – PChD)	6
Polish Peasant Party (*Polskie Stronnictwo Ludowe* – PSL)***	5
Stu Movement (*Ruch Stu*)	4
The Nonpartisan Bloc for Support of Reforms (*Bezpartyjny Blok Wspierania Reform* – BBWR)	3
Polish Union (*Zjednoczenie Polskie* – ZP)	3
Movement for the Commonwealth (*Ruch dla Rzeczpospolitej* – RdR)	2

*abbreviation AWS refers to these MPs who did not belong to any other party; this group was dominated by the activists of The Independent Self-Governing Trade Union "Solidarity" (*Niezależny Samorządny Związek Zawodowy "Solidarność"*)

**Into the group of KPN MPs I included also the MPs having connections with KPN but competing in the election as the distinct parties, e.g. Polish Western Union (*Polski Związek Zachodni*)

***Into the group of PSL MPs I included all AWS MPs having connections with various PSL fractions (i.a. Polish Peasant Party-*Peasant Agreement*: PSL-PL)

Source: Sejm of the Republic of Poland, Archives.

The final period of the 3rd term of the Polish Sejm turned out to be a catastrophe for AWS. Further groups of MPs tended to leave the club and some of the

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[6] Source: own – data and calculations. Cf. as well: Słomczyński et.al. 2008: p. 216).

groups would sign independent agreements. Admittedly, during its last period of existence AWS Parliamentary Club consisted of 134 MPs (at that time, the most numerous parliamentary club was created by a stable party, namely SLD), but the predominant role in its functioning was played by the so-called "modules", groups of MPs connected by party affiliations, created during the 3rd term. The tables below illustrate the structure of the Polish Sejm and separately the structure of AWSPC by the end of the 3rd term. Non-attached MPs, mostly the secessionists from different circles and clubs, are grouped here according to their party affiliation of the past (if they did not form any new political environment), or of the future – if they were co-creating PO or PiS at that time.

Table 3. Polish Sejm – end of the 3rd term

Club, parliamentary circle or informal group of MPs	No. of Mandates
Solidarity Electoral Action (AWS)	134
Democratic Left Alliance (SLD)	162
Union of Freedom (UW)	47
Polish Peasant Party (PSL)	26
Law and Justice (PiS)	18
The Conservative People's Party (SKL)	18
Polish Agreement (*Porozumienie Polskie* – PP)	5
The Alternative	4
Movement for the Reconstruction of Poland (*Ruch Odbudowy Polski* – ROP)	3
non-attached MPs (from PO)	16
non-attached MPs (from AWS)	17
non-attached MPs (German minority)	2
non-attached MPs (from SLD and PSL)	3
non-attached MPs (from ROP)	3
non-attached MPs (from UW)	2

Table 4. AWSPC – end of the 3rd term

Module affiliation	No. of Mandates
AWS Social Movement (*Ruch Społeczny AWS* – RS AWS)	87
Christian Democrats (*Chrześcijańscy Demokraci*)	19
Christian-National (*Chrześcijańsko-Narodowy*)	17
MPs who did not belong to any module	10
The Conservative People's Party (SKL)	1

Source: Own.

For both moments – the beginning and the end of the 3rd term of the Polish Sejm, I assigned values of the Shapley-Shubik power index referring to the Sejm structure with AWS as a single club (table 1 and 3) and taking into account the AWSPC members party affiliations (table 2 and 4). Comparing the index value of the AWSPC as a whole with the sum of index values of each club element treated separately enabled me to determine, which of the integrative and decentralizing factors had stronger influence on AWS MPs at the beginning and by the end of the 3rd term, which is illustrated by the table below, presenting Shapley-Shubik power index values for AWS Parliamentary Club and the total of values of the club components.

Table 5. Super- and subadditivity of AWS Parliamentary Club in different moments of the Polish Sejm the 3rd term

	Shapley-Shubik index value for AWSPC considered as a uniform bloc	**Shapley-Shubik index – the total of values calculated separately for each component of the AWSPC**
beginning by the	41.7%	35.7%
end of the 3rd term	21.3%	22.5%

As we can see, although at the beginning of the term AWSPC was more fragmented, it was a superadditive bloc. Both the structure of the Sejm and of the AWSPC itself accounted for cementing the new parliamentary club. By the end of the term, however, although less fragmented – AWSPC became subadditive. Apart from the immediate political factors (e.g. Marian Krzaklewski losing the presidential election, disintegration of the government coalition, etc.), configuration of the AWS encouraged many groups of AWS MPs to undertake independent action.

Example 3. Consolidation of PiS and PO in the Sejm of the 3rd term

As I have already mentioned, collapse of the AWS and weakening of the UW were accompanied by formation of two important parties: Civic Platform (PO) and Law and Justice (PiS). Formal registration of both parties was preceded by described new political environments – PO's association in January 2001 and PiS' local committees (March 2001). PiS Parliamentary Club, consisting of 18 MPs, appeared in the Sejm earlier then POPC. 16 PO MPs remained formally non-attached until the end of the 3rd term. It is also known, that as the end of the 3rd term was approaching, both of these groupings tended to integrate

increasingly within each other, so that from the potential blocs real and uniform political blocs emerged.

How to describe this situation using the formal theory? Each of the MPs groups had two strategies to choose from: "act together as a bloc" (for PO I marked this strategy as **PO**, for PiS – as **PiS**) or "each MP votes independently" (for PO I marked this strategy as **˜PO**, for PiS – as **˜PiS**).

I calculated the Shapley-Shubik power index value for 4 configurations:

(i)   PO and PiS act as consolidated blocs (they choose, respectively, strategies **PO** and **PiS**),
(ii) PO is consolidated, PiS uncoordinated (strategies, respectively, **PO** and **˜PiS**),
(iii) PO uncoordinated, PiS consolidated (strategies, respectively, **˜PO** and **PiS**),
(iv) PO uncoordinated, PiS uncoordinated (strategies, respectively, **˜PO** and **˜PiS**).

The variant in which one group – PO or PiS – adopted a strategy of uncoordinated voting, would put these group's MPs in a position of players of weights equal to 1.

The table below show the values of the Shapley-Shubik index of PO and PiS (or, in case of adopting a strategy of lack of coordination, the index values of the MPs forming a given potential bloc) for all analyzed options.

Table 6. Shapley-Shubik index values for PO and PiS depending on the choice of a strategy

		PiS strategies	
		PiS	˜PiS
PO	**PO**	**(3.37%,4.06%)**	(4.11%,3.6%)
strategies	**˜PO**	(3.06%,4.64%)	(3.72%,4.18%)

In the brackets, on the left hand side one can see the PO index values, and on the right hand side – index values for PiS. We can interpret them as a classical game-theoretical "players' payoffs". Let us start the analysis from the optimal strategy of PO: if PiS chooses the strategy of "playing in a consolidated bloc", then PO should also choose a consolidated strategy (3.37% > 3.06%). But if PiS will choose the no-coordination option, then PO will profit by choosing its own consolidation (4.11% > 3.72%). We can say that for PO the strategy "let's vote without coordination" was dominated by the strategy "let's act as a consolidated bloc" near the end of the 3rd term of the Sejm. It appears that in case of PiS, the dominating strategy was also consolidation. The part of the table which refers to these two dominating strategies, is in bold type. The result seems to be a credible explanation of the observations presented at the beginning of the case study.

5. Oceanic games in politics – stability of agreements. A second example of application

The problem of blocs' stability has also been explained in a more universal way – by applying the oceanic games theory. The Polish Sejm consists of 460 MPs, therefore it can be treated as a set composing of a few or several clubs as well as of a few hundred individual MPs acting independently and small groups of MPs trying to integrate. This illustrates particularly well the situation of the Polish Sejm near the end of its 3rd term of office. Just like Straffin (1982), let us study a weighted oceanic game with two large blocs-players X and Y, whose weight is, respectively, x and y, and an ocean of the weight α. Just like before, each of the major players has two strategies: "act as a consolidated, actual bloc" and "act as a group of non-organized single voters". Two cases should be considered:

a) The case considered by Straffin: both blocs do not exist in an organized form and the alternative for consolidation of both blocs (game $[1/2; x, y; \alpha]$) is the non-atomic game – one ocean of players. *De facto*, we study here the determinants of possible consolidation of the ocean's fragments. A proper example is the above-mentioned situation of PO and PiS environments at the beginning of their political career (end of the 3rd term of the Polish Sejm).

b) The recommended supplement of the model: there are two blocs of voters (x and y in the play $[1/2; x, y; \alpha]$) and in the case of a collapse of a given bloc – e.g. bloc x – the result then shall be the case described by the game $[1/2; y; x+ +\alpha]$, while in the case of bloc y's collapse – by the game $[1/2; x; y+\alpha]$. Such an approach will let us describe the determinants of possible disintegration of the existing blocs. An appropriate example may be the situation of the existing parliamentary clubs, e.g. AWS during the 3rd term of office of the Polish Sejm, or blocs of pledged delegates of Obama and Clinton during the 2008 primary election.

In the case of applying the consolidation strategies, power index values of both major players (or fragments of the ocean) will be described by the expressions presented at the end of section 2 of this article (respectively: φ_x and φ_y). But if the members of blocs X or Y choose the "let's vote independently" strategy, then their total power will equal power of a corresponding part of the ocean [7]. Depending on the situations describe above, it shall be:

a) A fraction of the votes of the disintegrated bloc-player (part of the ocean) in the non-atomic game, i.e. respectively $\Phi_x = x$ or $\Phi_y = y$;

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[7] Justification of this statement can be found, among others, in (Milnor, Shapley, 1961) and (Jasiński, 2009).

b) A fraction of the power of the ocean, which has been expanded by the disintegrated members of a given bloc. In the case of disintegration of player X the power of the ocean equals: $\Phi = 1 - \dfrac{y}{1-y}$. Therefore combined power of the disintegrated bloc X will be:

$$\Phi_x = \frac{x}{1-y}\left(1 - \frac{y}{1-y}\right) = \frac{(1-2y)\,x}{(1-y)^2}.$$

By the symmetry, combined power of the disintegrated bloc Y will equal:

$$\Phi_y = \frac{(1-2x)\,y}{(1-x)^2}.$$

So, blocs-players X and Y will gain on their own consolidation (they will be superadditive) when, respectively, $\varphi_x > \Phi_x$ and $\varphi_y > \Phi_y$. Figures 2 and 3 present solutions of both these inequalities for both situations.

Figure 2. Situation *a)* regions of consolidation and disintegration X and Y in the game $[1/2; \mathbf{x}, \mathbf{y}; \alpha]$
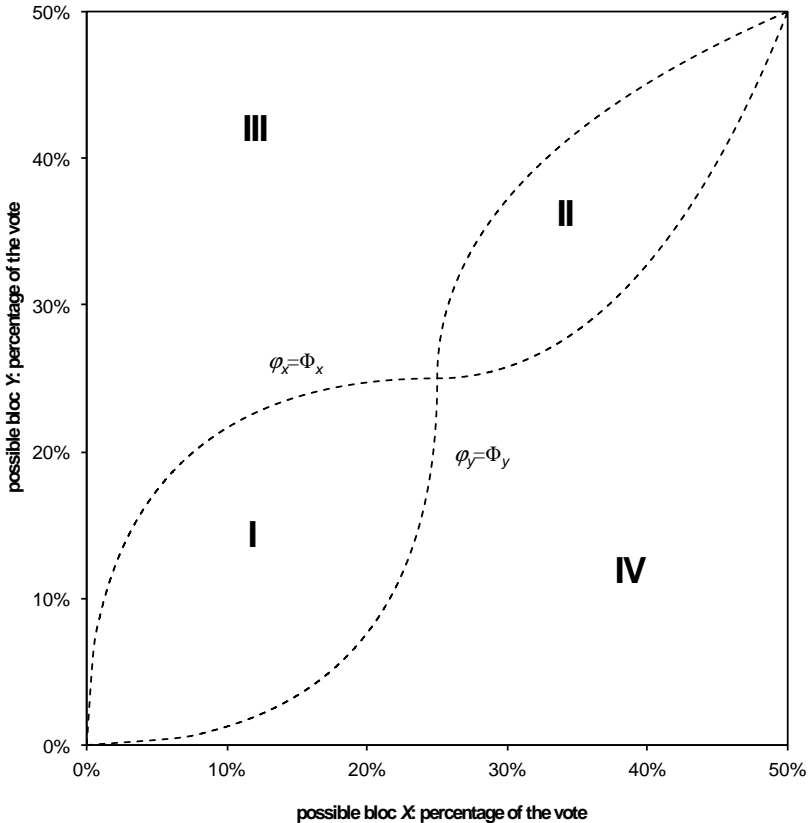
On both coordinates of the chart there are marked points showing the percentage of the votes won by players X and Y. The curves illustrate graphs of functions $\varphi_x = \Phi_x$ and $\varphi_y = \Phi_y$ for the situation described in point a), i.e. defining the conditions enabling the fractions of the ocean to integrate into blocs. Points of the chart represent different configurations of the weights of players X and Y. Here is a description of the chart's regions:

- In region I both players gain by the consolidation.
- In region II both players lose by the consolidation.
- In region III Y gains, while X loses by the consolidation.
- In region IV X gains, while Y loses by the consolidation.

As we can see in Figure 2, there are possible situations in which even large potential blocs do not have "purely strategic" motives for integration (e.g. area II) and conversely – in certain configurations also some smaller environments have motives for consolidations (e.g. area I).

Figure 3. Situation *b)* regions of consolidation and disintegration X and Y in the game $[1/2; \mathbf{x}, \mathbf{y}; \alpha]$
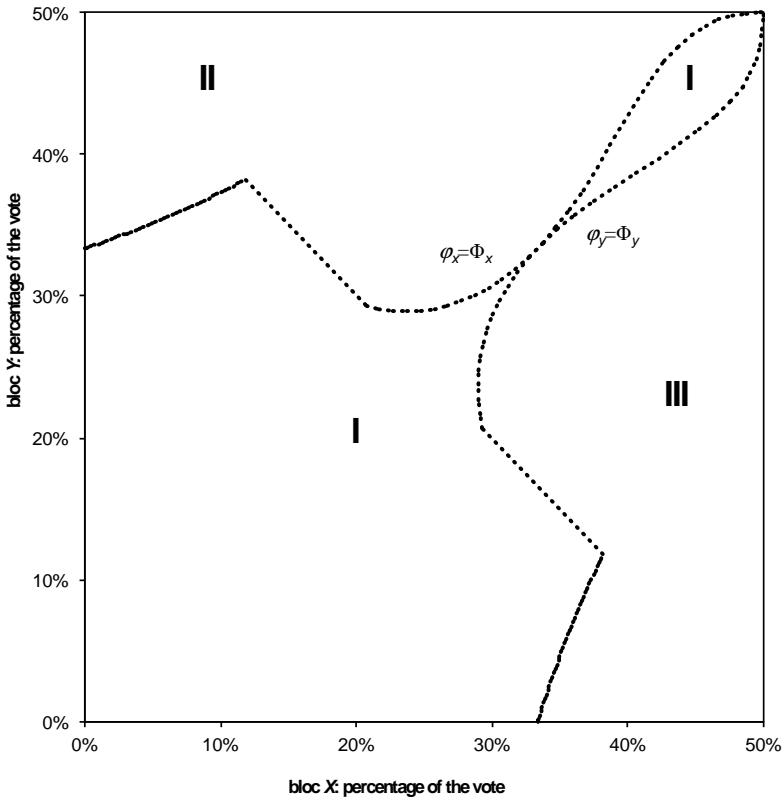
Figure 3 presents graphs of functions $\varphi_x = \Phi_x$ and $\varphi_y = \Phi_y$ for situation described in the point b), i.e. defining the conditions enabling the existing blocs to consolidate.

Here is a description of the areas presented in Figure 3:
- In region I both players gain by the consolidation.
- In region II Y gains, while X loses by the consolidation.
- In region III X gains, while Y loses by the consolidation.

Also here one can find several interesting interpretations, e.g. just a slight dominance of bloc X over bloc Y (e.g. 45% X and 40% Y) is a condition in which there are strategic motives for consolidation of bloc X and lack of such motives for bloc Y voters.

Figure 4. Application of this model to the description of consolidation followed by disintegration of the AWS during the 3rd term of office of the Polish Sejm
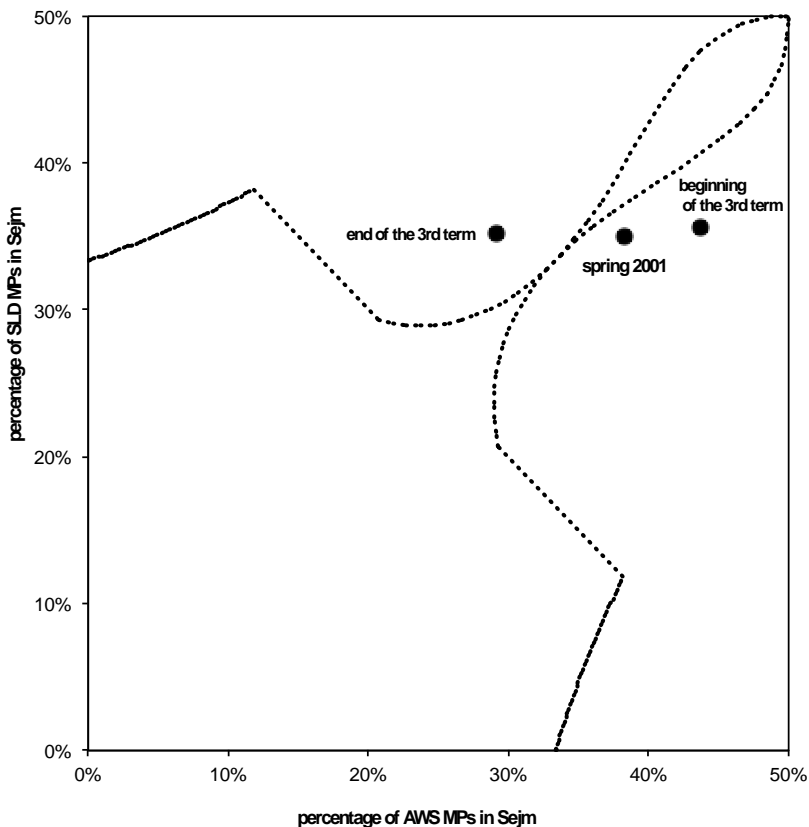
The following three Figures present the graphs discussed above, complemented by the results. As one will see, application of the described models shall allow for achieving easily interpretable explanations of the discussed phenomena.

Figure 4 presents a graph of b) type with marked fractions of AWS and SLD – the biggest clubs both during and at the end of the 3rd term of the Sejm (we study here the determinants of consolidation/disintegration of the existing blocs).

Figure 5 presents both graphs (of type a and b) with marked votes fractions of the PO and PiS from the period of their forming at the end of the 3rd term of the Sejm (we will use the type a model here) and votes fractions of the PO and PiS parliamentary clubs during the 6th and 7th terms of the Polish Sejm (type b model).

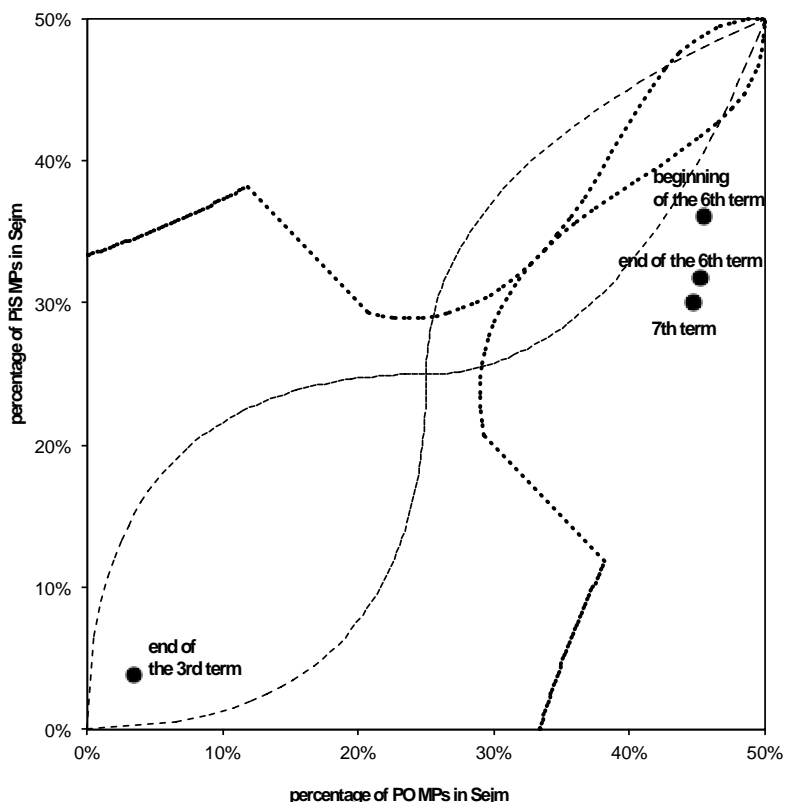**Figure 5. Application of this model to the description of the relation of PO to PiS**

Figure 6 presents a graph of type b) and a bandwagon curve with marked subsequent results obtained by Obama and Clinton in the primary election. Two groups of determinants were required for the flow of voters, committed to Clinton

(pledged delegates) to the bloc of Obama's supporters, to start: the consolidation of Obama's bloc combined with the disintegration of Clinton's bloc triggering the bandwagon effect. Two bigger black points represent the votes won by both competitors to the 6th and 15th of May.

Figure 6. Application of this model to the description of Clinton's bloc disintegration and flow of delegates to Obama's bloc in 2008
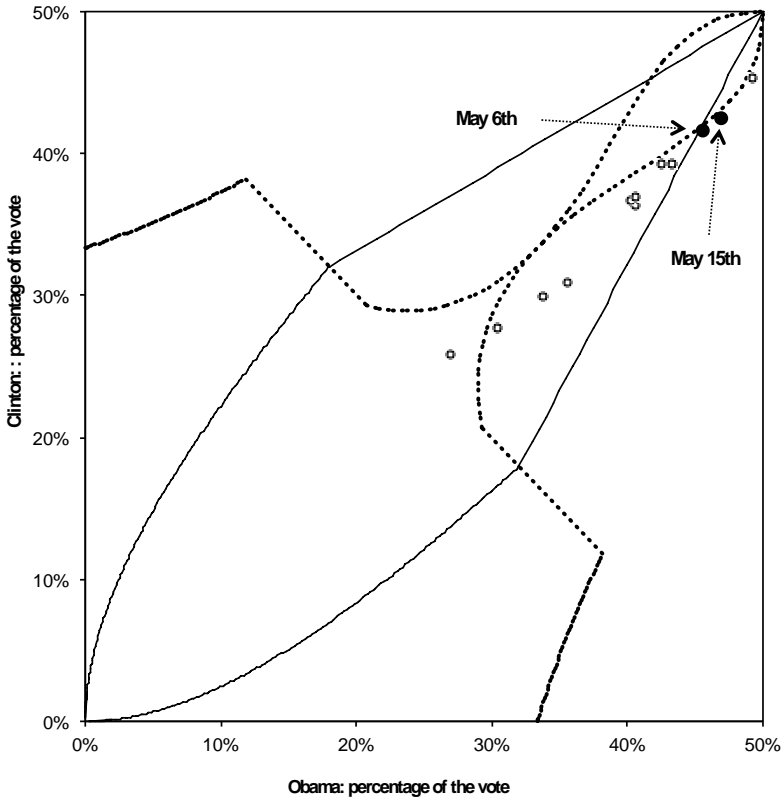
Particular points marked on the figures represent the following phenomena, discussed in this article:

- Figure 4. AWS and SLD at the beginning of the 3rd term – consolidation of AWSPC (201 mandates, i.e. 43.7% of the total mandate number), competing with SLDPC (164 mandates, i.e.35.7% of the total mandate number). Position of this point is in accordance with the earlier analyses showing that at the beginning of the 3rd term AWSPC MPs gained while voting as an organized and consistent bloc.
- Figure 4. AWS and SLD at the end of the 3rd term – disintegration of the AWSPC (134 mandates, i.e.29.1% of the total mandate number) competing

with SLDPC (164 mandates, i.e.35.7% of the total mandate number). Losing increasingly more MPs, AWSPC "shifted" its point into the area where it stopped to be profitable for its MPs to continue the strategy "let's act in unison".

- Figure 5. PiS and PO at the end of the 3rd term – consolidation of both PiS (18 mandates, i.e. 3.9% of the total mandate number), and PO (16 mandates, i.e. 3.5% of the total mandate number). Despite the weakness of the newly created party structures, PiS and PO MPs in the Sejm received strategic "encouragements" to consolidate their parliamentary environments.

- Figure 5. PiS and PO during the 6th and 7th term. During both terms the second largest parliamentary club (PiSPC) remains in a very disadvantageous configuration, which does not favour its consolidation. Perhaps this was the reason why during both terms PiS systematically initiated "closed ranks" actions, very strong ideologically and as such creating an additional "bond" cementing this party. On the contrary, POPC – appears to be a bloc being consolidated by the advantageous proportion of votes as compared with the competitor. These observations are confirmed by the classic analysis proving POPC's superadditivity in the context of possible conflict with the "conservatives" of Jarosław Gowin. The former Minister of Justice can count on about 40 people (that was the number of POPC MPs voting against the bill on "civil partnerships") The Shapley-Shubik index value of the POPC acting as a uniform bloc of 206 MPs equals 60%, while the total of the index values of both POPC components (166+40) in case of possible secession of the conservatives equals just under 45%.

- Figure 6. Clinton and Obama in 2008. The second highlighted point marked by the date May 15 presents the percentage of Obama's and Hillary Clinton's votes won to that day (respectively 46.9% and 42.5%). On that day a sudden increase of the number of Obama supporters started. Simultaneously, the bloc supporting Clinton started disintegrating, as the pledged delegates started to shift their support to Obama. As one can see, this point not only remains in the area of Obama's bloc consolidation and Clinton's bloc disintegration, but also in the area of the bandwagon effect. Just then both conditions determining the phenomenon described above were satisfied.

It is noticeable that the results of applying the oceanic games model confirmed all conclusions formulated before. Its advantage is – because of the assumption concerning continuum of players, being in fact a certain approximation of the reality (the Sejm of 460 MPs or the over 4,000 delegates of the electoral convention) – the simplicity of reasoning and its naturalness in the describing the reality of making collective decisions. In the situation when the voters have all the other voters in sight yet without distinguishing individuals, since they were

blending into a mass, assuming the continuum of the players instead of a group of individuals seems to be the proper approach.

6. Summary

In the article I presented a proposal to describe some of the political life manifestations by applying the models based on the game theory. I showed that different models, both based on classical power indices and on multi-player games, allow achievement of easily interpretable and coherent results, being in accordance with intuition.

The discussed models are of course limited in some ways. Analysis of the parliamentary games was performed in separation from any events taking place outside the parliament, like behaviour of the party structures or voters' preferences. These aspects could have a great influence on the situation in the Parliament. One can say about mutual influence of these two factors on each other. Definitely we should not neglect the outside aspect of the sudden changes in politics, e.g. the disintegration of the AWSPC at the end of the 3rd term, accompanied by collapse of the AWS local structures and many important activists abandoning the party. This of course, weakens the power of the empiric conclusions formulated as a result of the described approach. Resisting an urge to seek the relationship of deterministic character, one can still say about the value of the discussed models in the context of pointing out the significant and often dominating inside factors influencing the decision-making bodies.

In a case of great institutional stability of the studied institutions, it is possible to explain effectively or even predict phenomena taking place within large decision making assemblies, while concentrating exclusively on the inner configurations. It applies both to examining the processes within such institutions, like parliament or company's shareholders meeting, as well as to primary elections of naturally unstable character. In the first case better predictions of the decision games shall be achieved in the mid-term of office, i.e. during the period of time lacking any serious scandals or catastrophes, or – in case of companies – a time of no turbulences concerning the company's quotations (stability of PO described above could prove to be deceptive in case of mass social protests and propagandistic mistakes of the party). In the other case of the USA primary election, the key element appears to be the stable electoral procedures and professionalism, as well as rationality of the delegates: common knowledge of the possible consequences of the decisions made, making intentional choices based on their own preferences and clear interests (joining the camp of the future president on time provides one with real chances of participating in his administrative structures).

The effectiveness of the parliamentary games analysis would also increase if one took into consideration the ideological factor, often important for political actions. The models presented above are based on the symmetrical power index analysis. There are also available spatial generalizations (i.a. Shapley, 1977; Jasiński, 2003) of different indices and also applications of power indices of games with a priori unions (Carreras & Owen, 1988; Sosnowska, 1995). Also, it is possible to effectively reconstruct the ideological space within the political bodies (Jasiński, 2012). Referring to the non-symmetrical indices will allow for developing the described tools, which shall be a subject of a separate article.

Bibliography

[1] Aumann R.J., Shapley L.S. (1974), *Values of non-atomic games*, Princeton, New Jersey, Princeton University Press.

[2] Carreras F. and Owen G. (1988), *Evaluation in Catalonian Parliament 1980-84,* „Mathematical Social Science" 15, pp. 87-92.

[3] Ekes M. (2003), *General elections modelled with infinitely many voter*s, „Control and Cybernetics" No. 32, pp. 163-173.

[4] Falsenthal D.S., Machover M. (1995), *Postulates and paradoxes of relative voting power – a critical re-appraisal*, „Theory and Decision" 38(2): pp. 195-229.

[5] Jasiński M. (2000), *Is the biggerone always more powerful, or how to measure the power of the participants in collective decisions?* „Sociological Studies" No. 1-2, pp. 49-77.

[6] Jasiński M. (2003), *Ideological positionand the importance of participant in decision-making body*, „Sociological Studies" No. 1, pp. 139-174.

[7] Jasiński M. (2009), *Decisions in large groups – oceanic games in social sciences,* „Decisions" No.12, pp. 25-51.

[8] Jasiński M. (2012), *The ideological space based on political events*, „Decisions" No. 17, pp. 5-28.

[9] Kamiński M. (1998), *Strategic consequences of joining and parting: analysis of the polish party system consolidation in the years 1993-97*, „Sociological Studies" No. 3, pp. 19-47.

[10] Malawski M. (2000), *„Oceanic" probabilistic values*, Warsaw, Institute of Computer Science, Polish Academy of Sciences, Working paper.

[11] Milnor J.W., Shapley L.S. (1961), *Values of Large Games II: Oceanic Games,* RM-2649, Rand Corporation, Reprint „Mathematics of Operations Research" 3 (1978), No. 4, pp. 290-307.

[12] Shapiro N.Z., Shapley L.S. (1960), *Values of Large Games I: A Limit Theorem*, RM-2648, Rand Corporation, Reprint „Mathematics of Operations Research" 3 (1978), No. 1, pp. 1-9.

[13] Shapley L.S. (1961), *Values of Large Games III: A Corporation with Two Large Stockholders*, RM-2650-PR, Rand Corporation, Reprint as a part of Milnor, J.W., Shapley L.S. (1978), Values of Large Games II: Oceanic Games in: „Mathematics of Operations Research" 3 (1978), No. 4, pp. 290-307.

[14] Shapley L.S. (1977), *A comparison of power indices and a non-symmetric generalization*, RAND Paper P-5872, Rand Corporation Santa Monica.

[15] Słomczyński W., Poznański P., Harańczyk G., Sokołowski J.K. (2008), *Integrity of parliamentary groups* in: Sokołowski J., Poznański P. (ed.) *Chosen aspects of the Sejm activities in the years 1997-2007,* Cracow, AFM Printing House, pp. 203-235.

[16] Sosnowska H. (1995), *Shapley value of games with a priori unions as method of analysis of elections in Poland 1989–94,* „Works on mathematical economy" No. 1/EM/96, Warsaw, Warsaw School of Economics.

[17] Straffin P.D. (1977), *The Bandwagon Curve*, „American Journal of Political Science" 21, No. 4: pp. 695-709.

[18] Straffin P.D. (1982), *Power indices in politics.* in: *Political and Related Models,* Brams S.J., Lucas W., Straffin P.D. (ed.) New York: Springer Verlag, pp. 256-321.

[19] Wieczorek A. (2004), *Large Games with Only Small Players and Finite Strategy Sets*, „Applicationes Mathematicae" No. 31, pp. 79-96.

[20] Wieczorek A. (2005), *Large Games with Only Small Players and Strategy Sets in Euclidean Spaces,* „Applicationes Mathematicae" No. 32, pp. 183-193.

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Warunki stabilności porozumień, czyli od czego zależą przepływy graczy między koalicjami

Abstrakt

W artykule przedstawiam jedno z zastosowań modeli teoriogrowych do opisu i wyjaśnienia warunków stabilności tworzonych koalicji. Prezentowane podejście bazuje na analizie indeksów siły. Ich badanie pozwala na określanie subaddytywności bądź superaddytywności koalicji. Przedstawiam również zastosowanie modelu gier oceanicznych do badania stabilności porozumień w zgromadzeniach z dużą liczbą graczy. Prezentację ilustruję wynikami badania procesów decyzyjnych w Sejmie RP oraz w zgromadzeniu delegatów podczas prawyborów w Partii Demokratycznej w USA przed wyborami prezydenckimi w roku 2008. Uzyskane wyniki pozwalają na zgodne z intuicjami interpretacje dobrze wyjaśniające obserwowane zjawiska w zgromadzeniach decyzyjnych.

Słowa kluczowe: koalicja, indeks siły Shapleya-Shubika, ważona gra większości, gra oceaniczna, sub- i superaddytywność koalicji.

Author:

Mikołaj Jasiński, Chair Statistics, Demography and Mathematical Sociology, Institute of Sociology, University of Warsaw, ul. Karowa 18, 00-927 Warsaw, Poland,

e-mail: mikolaj.jasinski@is.uw.edu.pl